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Maxwell's equations in a 1D ferromagnetic medium : existence and uniqueness of strong solutions

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Abstract: In this paper we are interested in Maxwell's equations together with the Landau-Lifchitz-Gilbert law in order to model absorbing ferromagnetic materials in 1D. Using a fixed point theorem we establish existence and uniqueness of strong global solutions in suitable spaces, namely $H(\text{curl}\,;\mathbb{R})$ for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$, and $\left(L^2 \cap L^\infty\right)^3$ for the magnetization $\mathbf{M}$.

Key-words: Maxwell's equations - Landau-Lifchitz-Gilbert law - fixed point theorem - a priori estimates

(Résumé : tsvp)

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Electromagnétisme en milieu ferromagnétique 1D : existence et unicité des solutions fortes

Résumé : On s'intéresse dans ce rapport à un problème de Cauchy relatif à la modélisation dans le cas monodimensionnel de matériaux ferromagnétiques, problème qui couple aux équations de Maxwell la loi non linéaire de Landau-Lifchitz-Gilbert. Mettant en oeuvre une technique de point fixe, on établit l'existence et l'unicité de solutions fortes globales dans des espaces "naturels", à savoir $H(\mathrm{rot}, \mathbb{R})$ pour les champs électriques et magnétiques $\mathbf{E}$ et $\mathbf{H}$, et $\left(L^2 \cap L^\infty\right)^3$ pour l'aimantation $\mathbf{M}$.

Mots-clé : Equations de Maxwell - loi de Landau-Lifchitz-Gilbert - théorème de point fixe - estimations a priori

1 Position of the result. Notations and main results

In this article we are interested in the Cauchy problem (see for instance [3], [4], [5],...)

$$
(\mathcal{P}) \qquad \left\{
\begin{array}{llll}
\varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} - \text{curl } \mathbf{H} = 0 & & x \in \mathbb{R}\,,\ t>0 & (1.1) \\
\mu_0 \dfrac{\partial \mathbf{H}}{\partial t} + \text{curl } \mathbf{E} = -\mu_0 \dfrac{\partial \mathbf{M}}{\partial t} & & x \in \mathbb{R}\,,\ t>0 & (1.2) \\
\dfrac{\partial \mathbf{M}}{\partial t} = \gamma \mathbf{H}_T \times \mathbf{M} + \dfrac{\alpha}{|\mathbf{M}|} \mathbf{M} \times \dfrac{\partial \mathbf{M}}{\partial t} & & x \in \mathbb{R}\,,\ t>0 & (1.3) \\
\mathbf{H}_T = \mathbf{H} + \mathbf{H}_s - K P(\mathbf{M}) & & x \in \mathbb{R}\,,\ t>0 & (1.4) \\
\mathbf{H}(x,0) = \mathbf{H}_0(x) & & x \in \mathbb{R} & (1.5) \\
\mathbf{E}(x,0) = \mathbf{E}_0(x) & & x \in \mathbb{R} & (1.6) \\
\mathbf{M}(x,0) = \mathbf{M}_0(x) & & x \in \mathbb{R} & (1.7)
\end{array}
\right.
$$

The unknowns of the problem are the three vector fields $(\mathbf{E}, \mathbf{H}, \mathbf{M})$; the data, in addition to $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{M}_0)$, are

- $\varepsilon_0, \mu_0, \gamma$: real strictly positive coefficients
- $\alpha(x), K(x)$: positive functions of the space variable x, possibly equal to 0
- $P(\mathbf{M}) = \mathbf{M} - (\mathbf{p} \cdot \mathbf{M})\mathbf{p}$, $\forall \mathbf{M} \in \mathbb{R}^3$: orthogonal projection on the plane perpendicular to $\mathbf{p}$
- $\mathbf{p}(x), \mathbf{H}_s(x)$: vector fields in $\mathbb{R}^3$

with the following assumptions

$$
(\mathcal{H}) \qquad \left\{
\begin{array}{l}
\bullet\ (\alpha, K, \mathbf{p}) \in L^\infty \times L^\infty \times L^\infty \text{ and } \mathbf{p}(x) \text{ is unitary : } |\mathbf{p}(x)| = 1, \text{ a.e. } x \in \mathbb{R} \\
\bullet\ (\mathbf{E}_0, \mathbf{H}_0) \in H(\text{curl }; \mathbb{R}) \times H(\text{curl }; \mathbb{R}),\ \mathbf{H}_{0,x} \in L^\infty \\
\bullet\ \mathbf{M}_0 \in \left(L^2 \cap L^\infty\right)^3,\ \mathbf{H}_s \in \left(L^2 \cap L^\infty\right)^3
\end{array}
\right.
$$

where we have used the following notations :

- $\forall \mathbf{u} = (u_x, u_y, u_z)^t \in \mathcal{D}'(\mathbb{R})^3$, curl $\mathbf{u} = \left(0, -\dfrac{\partial u_z}{\partial x}, \dfrac{\partial u_y}{\partial x}\right)^t$
- $H(\text{curl }; \mathbb{R}) = \{\mathbf{u} \in \left(L^2\right)^3, \text{curl } \mathbf{u} \in \left(L^2\right)^3\} = L^2 \times H^1 \times H^1$

(L^2 holds for $L^2(\mathbb{R})$, H^1 holds for $H^1(\mathbb{R})$, and so on...)

If φ belongs to $L^p(\mathbb{R})^k$, with $1 \le p \le +\infty$ and $k \in N$, we shall denote by $\|\varphi\|_{L^p}$ its L^p-norm. Of course $H(\text{curl }; \mathbb{R})$ is an Hilbert space equipped with the norm :

$$\|u\|^2_{\text{curl}} = \|u\|^2_{L^2} + \|\text{curl } u\|^2_{L^2} \tag{1.8}$$

In the sequel, it will also be useful to introduce the Banach space :

$$V = \left\{ (\mathbf{E}, \mathbf{H}, \mathbf{M}) \in H(\text{curl }) \times H(\text{curl }) \times \left(L^2 \cap L^\infty\right)^3 \,/\, H_x \in L^\infty(\mathbb{R}) \right\} \tag{1.9}$$

equipped with the norm :

$$\|(\mathbf{E}, \mathbf{H}, \mathbf{M})\|_V = \|\mathbf{E}\|_{\text{curl}} + \|\mathbf{H}\|_{\text{curl}} + \|\mathbf{M}\|_{L^2} + \|\mathbf{M}\|_{L^\infty} + \|H_x\|_{L^\infty} \tag{1.10}$$

Note that our assumptions on the initial data are nothing but

$$(\mathbf{E}_0, \mathbf{H}_0, \mathbf{M}_0) \in V \tag{1.11}$$

We can now give the definition of a strong solution of (P).

Definition 1 *Let $0 < T \leq +\infty$, we say that* $(\mathbf{E},\mathbf{H},\mathbf{M})$ *is a strong solution of (P) in* $[0,T]$ *if and only if :*

$$(i)\qquad \begin{cases}(\mathbf{E},\mathbf{H}) \in C^0\left(0,T;H(curl\,)\right)\cap C^1\left(0,T;(L^2)^3\right)\\ \mathbf{M} \in C^1\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^2\left(0,T;(L^2)^3\right)\end{cases}$$

$$(ii)\qquad \begin{cases}\textit{Equations (1.1) and (1.2) hold in } C^0\left(0,T;(L^2)^3\right)\\ \textit{Equations (1.3) and (1.4) hold in } C^0\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^1\left(0,T;(L^2)^3\right)\\ \textit{Equations (1.5) and (1.6) hold in } H(curl\,;\mathbb{R})\\ \textit{Equation (1.7) holds in } \left(L^2\cap L^\infty\right)^3\end{cases}$$

Remark 1 *The projection along the x-axis of equations (1.1) and (1.2) respectively give :*

$$\begin{cases}\dfrac{\partial}{\partial t}(H_x+M_x) & = & 0\\ \dfrac{\partial}{\partial t}(E_x) & = & 0\end{cases} \tag{1.12}$$

In particular $E_x = \mathbf{E}_{0,x}$ and $H_x + M_x = \mathbf{H}_{0,x} + \mathbf{M}_{0,x}$; we immediately deduce that H_x has the regularity :

$$H_x \in C^1\left(0,T;(L^2\cap L^\infty)^3\right) \tag{1.13}$$

Remark 2 *A solution on* $[0,+\infty)$ *– i.e.* $T = +\infty$ *– is called a global solution.*

We can now state the main theorem of this paper :

Theorem 1 *Under the assumptions* $(\mathcal{H})$, *the Cauchy problem* $(\mathcal{P})$ *admits a unique global solution* $(\mathbf{E},\mathbf{H},\mathbf{M})$, *which more over satisfies :*

$$|\mathbf{M}(x,t)| = |\mathbf{M}_0(x)| \quad a.e. x\in\mathbb{R}, \forall t\geq 0 \tag{1.14}$$

$$\frac{d}{dt}\mathcal{E}(\mathbf{E},\mathbf{H},\mathbf{M}) + \int_{\mathbb{R}}\frac{\alpha}{|\mathbf{M}|}\left|\frac{\partial\mathbf{M}}{\partial t}\right|^2 dx = 0 \tag{1.15}$$

when $\mathcal{E}(\mathbf{E},\mathbf{H},\mathbf{M})$ *denotes the electromagnetic energy defined by :*

$$\mathcal{E}(\mathbf{E},\mathbf{H},\mathbf{M}) = \frac{1}{2}\int_{\mathbb{R}}\left[\varepsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2 + \mu_0 K|P(\mathbf{M})|^2 + \mu_0|\mathbf{H}_s - \mathbf{M}|^2\right] dx \tag{1.16}$$

Remark 3 *Note that for any strong solution, the electromagnetic energy is a function of class* C^1 *with respect to time.*

Remark 4 *One has to make precise the sense of the integral* $\displaystyle\int_{\mathbb{R}}\frac{\alpha}{|\mathbf{M}|}\left|\frac{\partial\mathbf{M}}{\partial t}\right|^2 dx$. *In fact, from (1.3), it is easy to deduce (see section 2) that* $\displaystyle\left|\frac{\partial\mathbf{M}}{\partial t}\right|^2 = \frac{\gamma^2}{1+\alpha^2}|\mathbf{H}_T\times\mathbf{M}|^2$ *so that*

$$\int_{\mathbb{R}}\frac{\alpha}{|\mathbf{M}|}\left|\frac{\partial\mathbf{M}}{\partial t}\right|^2 dx = \gamma^2\int_{\mathbb{R}}\frac{\alpha}{1+\alpha^2}\frac{|\mathbf{H}_T\times\mathbf{M}|^2}{|\mathbf{M}|}dx \tag{1.17}$$

which makes sense since the function $M \mapsto \dfrac{|\mathbf{H}_T\times\mathbf{M}|^2}{|\mathbf{M}|}$ *can be continuously extended by 0 for* $\mathbf{M} = 0$. *Note that we have the estimate :*

$$\int_{\mathbb{R}}\frac{\alpha}{|\mathbf{M}|}\left|\frac{\partial\mathbf{M}}{\partial t}\right|^2 dx \leq \gamma^2\int_{\mathbb{R}}\frac{\alpha}{1+\alpha^2}|\mathbf{M}||\mathbf{H}_T|^2\,dx \tag{1.18}$$

which makes sense since one easily checks that for any time $\mathbf{M}\in(L^\infty)^3$ *and* $\mathbf{H}_T\in\left(L^2\right)^3$.

Remark 5 *For the theory,* $P(\mathbf{M})$ *can be replaced by any symmetric positive operator in* $\mathbb{R}^3$, *provides that its norm in* $\mathcal{L}\left(\mathbb{R}^3\right)$ *is uniformly bounded in space (we may assume that* $\mathbf{p}$ *depends on* x).

Identities (1.14) and (1.15) provide uniform estimates with respect to time about L^2 norms of $\mathbf{E}$ and $\mathbf{H}$ and $L^2 \cap L^\infty$ norm of $\mathbf{M}$. It is also interesting to have some information about the behaviour of the derivatives of $\mathbf{E}$ and $\mathbf{H}$. This is the object of the following theorem – whose proof is delayed to section 3 – which gives a polynomial bound in time for the L^2 norm of these quantities.

Theorem 2 *The unique global solution* $(\mathbf{E}, \mathbf{H}, \mathbf{M})$ *to Cauchy problem* $(\mathcal{P})$ *satisfies*

$$\begin{cases} \|curl\, \mathbf{E}(\cdot, t)\|_{L^2} & \leq \; C\,(1+t^2) \\ \|curl\, \mathbf{H}(\cdot, t)\|_{L^2} & \leq \; C\,(1+t^2) \end{cases}$$

and

$$\begin{cases} \left\| \dfrac{\partial \mathbf{E}}{\partial t}(\cdot, t) \right\|_{L^2} & \leq \; C\,(1+t^2) \\ \left\| \dfrac{\partial \mathbf{H}}{\partial t}(\cdot, t) \right\|_{L^2} & \leq \; C\,(1+t^2) \end{cases}$$

where C *is a positive constant depending only on intial data* $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{M}_0)$ *and on constants* ε_0, μ_0, α, γ, K *and* $\mathbf{H}_s$.

2 Proof of theorem 1

The principle of the proof follows the lines of the proof of existence and uniqueness theorems for ordinary differential equations ([2],[1]) :

(i) We first prove the existence and uniqueness of a local solution and give a lower bound for the time of existence as a function of the size of the initial data in appropriate norms. This result also provides the uniqueness result. Its proof is based on a standard fixed point argument and is the object of subsection 2.1.

(ii) We define the maximal solution and give an alternative which in particular gives a necessary and sufficient condition for blow-up in finite time (subsection 2.2).

(iii) We get a priori estimates which allow to prove that the maximal solution is in fact a global solution (subsection 2.3).

The different steps of the proof make use of some technical lemmas whose proof is delayed, for the sake of simplicity, to section 4.

Remark 6 *We will often use in the forthcoming computations the following classical result :*

$$\forall u \in H^1(\mathbb{R}), \qquad \|u\|_{L^\infty}^2 \;\leq\; 2\,\|u\|_{L^2} \left\| \frac{\partial u}{\partial x} \right\|_{L^2} \;\leq\; \|u\|_{H^1}^2$$

2.1 A local existence and uniqueness result

Theorem 3 *Under assumptions of theorem 1, there exists some* $T = T(\mathbf{E}, \mathbf{H}, \mathbf{M}) > 0$ *such that* $(\mathcal{P})$ *admits a unique strong solution on* $[0, T]$. *More over for any constant* $R > 0$, *there exists* $T_R > 0$ *such that :*

$$(\,\|(\mathbf{E}_0, \mathbf{H}_0, \mathbf{M}_0)\|_V \;\leq\; R\,) \quad \Longrightarrow \quad (\,T(\mathbf{E}_0, \mathbf{H}_0, \mathbf{M}_0) \;\geq\; T_R\,) \tag{2.1}$$

Remark 7 *The second part of the theorem mainly expresses that that the time of existence of the solution remains bounded from below when the initial data remain bounded in the space* V. *This will be essential for the proof of the global existence of subsection 2.3.*

Our proof of theorem 3 is analogous to proof of theorems for ordinary differential equations. It consists in applying a fixed point theorem to some function Φ acting on the space :

$$W(0,T) \;=\; \left\{u \in C^1\left(0,T;(L^2)^3\right) \cap C^0\left(0,T;H(\mathrm{curl}\,)\right)\,/\; u_x \in C^0\left(0,T;L^\infty\right)\right\} \tag{2.2}$$

which is a Banach space with the norm

$$\|u\|_{W(0,T)} = \sup_{t\leq T}\left\{\|\mathbf{u}(\cdot,t)\|_{L^2} + \left\|\frac{\partial \mathbf{u}}{\partial t}(\cdot,t)\right\|_{L^2} + \|\text{curl }u(\cdot,t)\|_{L^2} + \|u_x(\cdot,t)\|_{L^\infty}\right\} \tag{2.3}$$

The argument of the function Φ will be the magnetic field $\mathbf{H}$ and T will be appropriately chosen in order that Φ be a contraction in $W(0,T)$.

Let us now describe in detail how we construct the application Φ. Let $\mathbf{H}$ be a element of $W(0,T)$, we define $\mathbf{M}\in C^1\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^2\left(0,T;(L^2)^3\right)$ as the solution of

$$\left\{\begin{aligned} \frac{\partial \mathbf{M}}{\partial t} &= \gamma \mathbf{H}_T\times\mathbf{M} + \frac{\alpha}{|\mathbf{M}|}\mathbf{M}\times\frac{\partial \mathbf{M}}{\partial t}\\ \mathbf{M}(x,0) &= \mathbf{M}_0\\ \mathbf{H}_T &= \mathbf{H}+\mathbf{H}_s - KP(\mathbf{M}) \end{aligned}\right. \tag{2.4}$$

The existence and uniqueness of $\mathbf{M}$ is ensured by the following theorem that we shall prove in section 4.

Theorem 4 *For any $\mathbf{H}$ in $W(0,T)$, problem (2.4) admits a unique solution*

$$\mathbf{M}\in C^1\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^2\left(0,T;(L^2)^3\right)$$

Setting $\mathbf{M}=\mathcal{M}(\mathbf{H})$ we define a mapping from $W(0,T)$ into $C^1\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^2\left(0,T;(L^2)^3\right)$ wich more over satisfies :

$$a.e.\ x\in\mathbb{R} \qquad |\mathbf{M}(x,t)| = |\mathbf{M}_0(x)| \tag{2.5}$$

Remark 8 *It appears to be an application of the Cauchy-Lipschitz theorem in an appropriate framework. We give the details in the subsection 4.2. Concerning (2.5), it appears that a very simple consequence of the properties of the cross product is :*

$$\frac{\partial \mathbf{M}}{\partial t}(x,t)\cdot\mathbf{M}(x,t) = 0 \tag{2.6}$$

One then concludes.

From the knowledge of $\mathbf{M}\in C^1\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^2\left(0,T;(L^2)^3\right)$, we define $(\mathbf{E},\mathbf{H}^*)$ as the unique solution of the linear problem

$$\left\{\begin{aligned} &\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} - \text{curl }\mathbf{H}^* = 0 && x\in\mathbb{R},\ t>0\\ &\mu_0\frac{\partial \mathbf{H}^*}{\partial t} + \text{curl }\mathbf{E} = -\mu_0\frac{\partial \mathbf{M}}{\partial t} && x\in\mathbb{R},\ t>0\\ &\mathbf{E}(x,0) = \mathbf{E}_0(x) && x\in\mathbb{R}\\ &\mathbf{H}^*(x,0) = \mathbf{H}_0(x) && x\in\mathbb{R} \end{aligned}\right. \tag{2.7}$$

The existence and uniqueness of $(\mathbf{E},\mathbf{H}^*)$ is ensured by the analysis of linear Maxwell's equations. More precisely we have the following theorem

Theorem 5 *For any $\mathbf{M}\in C^1\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^2\left(0,T;(L^2)^3\right)$, if $(\mathbf{E}_0,\mathbf{H}_0)$ satisfy the second condition of $(\mathcal{H})$, problem (2.7) admits a unique solution $(\mathbf{E},\mathbf{H}^*)$ in $C^1\left(0,T;(L^2)^3\right)\cap C^0\left(0,T;H(curl\,)\right)$. Setting $\mathbf{E}=\mathcal{F}(\mathbf{M})$ and $\mathbf{H}^*=\mathcal{H}(\mathbf{M})$, we define two linear and continuous mappings from $C^1\left(0,T;(L^2\cap L^\infty)^3\right)\cap C^2\left(0,T;(L^2)^3\right)$ into $C^1\left(0,T;(L^2)^3\right)\cap C^0\left(0,T;H(curl\,)\right)$ and $W(0,T)$ respectively. Moreover we have the estimate*

$$\|\mathbf{H}^*\|_{W(0,T)} \leq \tau R+\sigma R^2 + \left\|\frac{\partial \mathbf{M}}{\partial t}\right\|_{L^1(0,T;L^2)} + (1+\sqrt{\varepsilon_0\mu_0})\left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}\right\|_{L^1(0,T;L^2)} \tag{2.8}$$

– where τ and σ denote constants which only depend on ε_0, μ_0, α, γ, K and $\mathbf{H}_s$ – provided that $\mathbf{M}=\mathcal{M}(\mathbf{H})$ and that $\|(\mathbf{E}_0,\mathbf{H}_0\mathbf{M}_0)\|_V\leq R$.

We can now define an application in $W(0,T)$ by :

$$\Phi \;=\; \mathcal{H} \circ \mathcal{M} \tag{2.9}$$

In other words

$$\mathbf{H}^* \;=\; \Phi(\mathbf{H}) \tag{2.10}$$

The research of a local strong solution of $(\mathcal{P})$ on $[0,T]$ is equivalent to the research of a fixed point of Φ in the sense of the following result (whose proof is trivial and left to the reader) :

Lemma 1 **(i)** *If* $(\mathbf{E},\mathbf{H},\mathbf{M})$ *is a strong solution of* $(\mathcal{P})$ *on* $[0,T]$, $\mathbf{H}$ *is a fixed point of* Φ.

(ii) *If* $\mathbf{H}$ *is a fixed point of* Φ, *then* $(\mathbf{E}=\mathcal{F}\circ\mathcal{M}(\mathbf{H})\,,\,\mathbf{H}\,,\,\mathbf{M}=\mathcal{M}(\mathbf{H}))$ *is a strong solution of* $(\mathcal{P})$ *on* $[0,T]$.

Our goal is to apply the contraction mapping theorem to Φ in some closed ball of $W(0,T)$. Let us consider :

$$E_R(T) \;=\; \left\{\mathbf{H}\in W(0,T)\,/\,\|\mathbf{H}\|_{W(0,T)} \le 1+\tau R+\sigma R^2\right\} \tag{2.11}$$

(where τ and σ denote constants which only depend on ε_0, μ_0, α, γ, K and $\mathbf{H}_s$).

The first step of the proof consists in proving that for T small enough, Φ maps $E_R(T)$ into itself.

Lemma 2 *Assuming that* $\|(\mathbf{E}_0,\mathbf{H}_0\mathbf{M}_0)\|_V \le R$, *there exists* $T_R^{(1)}>0$ *such that*

$$\forall T < T_R^{(1)}\,,\quad \Phi(E_R(T)) \subset E_R(T)$$

Before proving lemma (2), we first need a property of application $\mathcal{M}$:

Lemma 3 *Assuming that* $\|(\mathbf{E}_0,\mathbf{H}_0\mathbf{M}_0)\|_V \le R$ *and that* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *belong to* $E_R(T)$, *there exists positive constants* β_R, B_R, C_R *and* D_R *which only depend on* ε_0, μ_0, α, γ, K, $\mathbf{H}_s$ *and* R *such that the two following Lipschitz-continuity estimates hold :*

$$\left\|\frac{\partial \mathbf{M}_1}{\partial t}-\frac{\partial \mathbf{M}_2}{\partial t}\right\|_{L^1(0,T;L^2)} \le\; T\left(B_R\left(\frac{e^{\beta_R T}-1}{\beta_R}\right)+\gamma R\right)\|\mathbf{H}_1-\mathbf{H}_2\|_{W(0,T)} \tag{2.12}$$

$$\left\|\frac{\partial^2 \mathbf{M}_1}{\partial t^2}-\frac{\partial^2 \mathbf{M}_2}{\partial t^2}\right\|_{L^1(0,T;L^2)} \le\; T\left(C_R\left(\frac{e^{\beta_R T}-1}{\beta_R}\right)+D_R\right)\|\mathbf{H}_1-\mathbf{H}_2\|_{W(0,T)} \tag{2.13}$$

Proof. See subsection 4.2. □

Remark 9 *The important point in the estimates (2.12) and (2.13) is the fact that the constants in front of* $\|\mathbf{H}_1-\mathbf{H}_2\|_{W(0,T)}$ *tend to* 0 *when* T *vanishes.*

Proof of lemma 2. Let us set $\mathbf{H}^*=\Phi(\mathbf{H})$, where $\mathbf{H}$ belongs to $E_R(T)$. According to theorem 5, we have, with $\mathbf{M}=\mathcal{M}(\mathbf{H})$:

$$\|\mathbf{H}^*\|_{W(0,T)} \le \tau R+\sigma R^2+\left\|\frac{\partial \mathbf{M}}{\partial t}\right\|_{L^1(0,T;L^2)}+(1+\sqrt{\varepsilon_0\mu_0})\left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}\right\|_{L^1(0,T;L^2)} \tag{2.14}$$

Let us get estimates on $\dfrac{\partial \mathbf{M}}{\partial t}$ and $\dfrac{\partial^2 \mathbf{M}}{\partial t^2}$. As

$$\frac{\partial \mathbf{M}}{\partial t}-\frac{\alpha}{|\mathbf{M}|}\mathbf{M}\times\frac{\partial \mathbf{M}}{\partial t} \;=\; \gamma\mathbf{H}_T\times\mathbf{M} \tag{2.15}$$

We deduce that, a.e. $x\in\mathbb{R}$,

$$|\mathbf{M}(x,t)|^2 \;=\; |\mathbf{M}_0(x)|^2 \tag{2.16}$$

$$\left|\frac{\partial \mathbf{M}}{\partial t}(x,t)\right|^2 \;=\; \frac{\gamma^2}{1+\alpha^2}\,|(\mathbf{H}_T\times\mathbf{M})\,(x,t)|^2 \tag{2.17}$$

Equation (2.16) is obtained by taking the scalar product of equation (2.14) by $\mathbf{M}$ and equation (2.17) is obtained by squaring equation (2.14). On the other hand, from (1.4) we deduce :

$$|\mathbf{H}_T(x,t)| \leq |\mathbf{H}_s(x)| + |\mathbf{H}(x,t)| + |K(x)||\mathbf{M}(x,t)| \tag{2.18}$$

Taking into account (2.16), we get

$$\|\mathbf{H}_T(\cdot,t)\|_{L^2} \leq \|\mathbf{H}_s\|_{L^2} + \|\mathbf{H}(\cdot,t)\|_{L^2} + \|K\|_{L^\infty} \|\mathbf{M}_0\|_{L^2} \tag{2.19}$$

Then, by (2.17), we obtain

$$\left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} \leq \gamma \|\mathbf{M}_0\|_{L^\infty} \left(\|\mathbf{H}_s\|_{L^2} + \|\mathbf{H}(\cdot,t)\|_{L^2} + \|K\|_{L^\infty} \|\mathbf{M}_0\|_{L^2}\right) \tag{2.20}$$

which yields

$$\left\|\frac{\partial \mathbf{M}}{\partial t}\right\|_{L^1(0,T;L^2)} \leq \gamma R \left(\|\mathbf{H}_s\|_{L^2} + \|\mathbf{H}\|_{W(0,T)} + \|K\|_{L^\infty} R\right) T \tag{2.21}$$

that is to say, since $\mathbf{H}$ belongs to $E_R(T)$:

$$\left\|\frac{\partial \mathbf{M}}{\partial t}\right\|_{L^1(0,T;L^2)} \leq \gamma R \left(\nu^* + \tau^* R + \sigma^* R^2\right) T \tag{2.22}$$

where $\nu^* = 1 + \|\mathbf{H}_s\|_{L^2}$, $\tau^* = \tau + \|K\|_{L^\infty}$ and $\sigma^* = \sigma$ are positive constants depending only on ε_0, μ_0, α, γ, K and $\mathbf{H}_s$. Differentiating (2.14) with respect to time leads to :

$$\frac{\partial^2 \mathbf{M}}{\partial t^2} = \gamma \left(\mathbf{H}_T \times \frac{\partial \mathbf{M}}{\partial t} + \frac{\partial \mathbf{H}_T}{\partial t} \times \mathbf{M}\right) + \frac{\alpha}{|\mathbf{M}|}\mathbf{M} \times \frac{\partial^2 \mathbf{M}}{\partial t^2} \tag{2.23}$$

from which we get

$$\begin{aligned} \left|\frac{\partial^2 \mathbf{M}}{\partial t^2}(x,t)\right| &= \gamma \left|\left(\mathbf{H}_T \times \frac{\partial \mathbf{M}}{\partial t} + \frac{\partial \mathbf{H}_T}{\partial t} \times \mathbf{M}\right)(x,t)\right| \\ &\leq \gamma \left|\mathbf{H}_T(x,t)\right| \left|\frac{\partial \mathbf{M}}{\partial t}(x,t)\right| + \left|\frac{\partial \mathbf{H}_T}{\partial t}(x,t)\right| |\mathbf{M}(x,t)| \end{aligned} \tag{2.24}$$

From (2.18), we get :

$$\|\mathbf{H}_T(\cdot,t)\|_{L^\infty} \leq \|\mathbf{H}_s\|_{L^\infty} + \|\mathbf{H}(\cdot,t)\|_{L^\infty} + \|K\|_{L^\infty} \|\mathbf{M}_0\|_{L^\infty} \tag{2.25}$$

On the other hand, differentiating (1.4) with respect to time gives :

$$\frac{\partial \mathbf{H}_T}{\partial t} = \frac{\partial \mathbf{H}}{\partial t} - KP\left(\frac{\partial \mathbf{M}}{\partial t}\right) \tag{2.26}$$

and then

$$\left\|\frac{\partial \mathbf{H}_T}{\partial t}(\cdot,t)\right\|_{L^2} \leq \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2} + \|K\|_{L^\infty} \left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} \tag{2.27}$$

Going back to (2.24), we get :

$$\left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}(\cdot,t)\right\|_{L^2} \leq \gamma \|\mathbf{H}_T(\cdot,t)\|_{L^\infty} \left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} + \gamma \left\|\frac{\partial \mathbf{H}_T}{\partial t}(\cdot,t)\right\|_{L^2} \|\mathbf{M}_0\|_{L^\infty} \tag{2.28}$$

that is to say, thanks to (2.25) and (2.27)

$$\begin{aligned} \left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}(\cdot,t)\right\|_{L^2} &\leq \gamma \left(\|\mathbf{H}_s\|_{L^\infty} + \|\mathbf{H}\|_{L^\infty} + \|K\|_{L^\infty} R\right) \left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} \\ &+ \gamma \left(\left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2} + \|K\|_{L^\infty} \left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2}\right) R \end{aligned} \tag{2.29}$$

As

$$\begin{aligned} \|\mathbf{H}(\cdot,t)\|_{L^\infty} &\le \|H_x(\cdot,t)\|_{L^\infty} + \|H_y(\cdot,t)\|_{L^\infty} + \|H_z(\cdot,t)\|_{L^\infty} \\ &\le \|H_x(\cdot,t)\|_{L^\infty} + \|H_y(\cdot,t)\|_{H^1} + \|H_z(\cdot,t)\|_{H^1} \end{aligned}$$

we deduce that

$$\forall t \le T \qquad \|\mathbf{H}(\cdot,t)\|_{L^\infty} \le \|\mathbf{H}\|_{W(0,T)} \tag{2.30}$$

Therefore, using (2.22) and (2.34), (2.29) can be written

$$\begin{aligned} \left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}(\cdot,t)\right\|_{L^2} &\le \gamma\left[\|K\|_{L^\infty}\left(\|\mathbf{H}_s\|_{L^\infty} + \|\mathbf{H}\|_{W(0,T)} + \|K\|_{L^\infty} R\right) + \|\mathbf{H}\|_{W(0,T)}\right] R \\ &+\gamma^2\left[\|\mathbf{H}_s\|_{L^\infty} + \|\mathbf{H}\|_{W(0,T)} + \|K\|_{L^\infty} R\right]\left[\|\mathbf{H}_s\|_{L^2} + \|\mathbf{H}\|_{W(0,T)} + \|K\|_{L^\infty} R\right] \end{aligned} \tag{2.31}$$

As $\mathbf{H} \in E_R(T)$, it is not difficult to see that

$$\left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}\right\|_{L^1(0,T;L^2)} \le P_4(R)\,T \tag{2.32}$$

where $P_4(R)$ is a polynomial of degree 4 with respect to R, whose coefficients only on ε_0, μ_0, α, γ, K and $\mathbf{H}_s$. Regrouping (2.21) and (2.32), we get :

$$\left\|\frac{\partial \mathbf{M}}{\partial t}\right\|_{L^1(0,T;L^2)} + (1+\sqrt{\varepsilon_0\mu_0})\left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}\right\|_{L^1(0,T;L^2)} \le P_4^*(R)\,T \tag{2.33}$$

where $P_4^*(R)$ is a polynomial of degree 4, depending only on $(\varepsilon_0, \mu_0, \alpha, \gamma, K, \mathbf{H}_s)$, with positive coefficients. Plugging (2.33) into (2.15), we get :

$$\|\mathbf{H}^*\|_{W(0,T)} \le P_4^*(R) + \tau R + \sigma R^2 \tag{2.34}$$

which gives the result provided that :

$$T \le T_R^{(1)} = \frac{1}{P_4^*(R)} \tag{2.35}$$

□

Lemma 4 *Assuming that* $\|(\mathbf{E}_0, \mathbf{H}_0\mathbf{M}_0)\|_V \le R$*, there exists* $T_R^{(1)} > 0$ *such that,* $\forall T < T_R^{(2)}$*,* $\Phi_{|E_R(T)}$ *is a strict contraction.*

Proof. Let us set $\mathbf{H}_1^* = \Phi(\mathbf{H}_1)$ and $\mathbf{H}_2^* = \Phi(\mathbf{H}_2)$, $\mathbf{M}_1 = \mathcal{M}(\mathbf{H}_1)$ and $\mathbf{M}_2 = \mathcal{M}(\mathbf{H}_2)$. First according to theorem 5 and taking into account the linear character of Maxwell's equations, we have :

$$\|\mathbf{H}_1^* - \mathbf{H}_2^*\|_{W(0,T)} \le \left\|\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t}\right\|_{L^1(0,T;L^2)} + (1+\sqrt{\varepsilon_0\mu_0})\left\|\frac{\partial^2 \mathbf{M}_1}{\partial t^2} - \frac{\partial^2 \mathbf{M}_2}{\partial t^2}\right\|_{L^1(0,T;L^2)} \tag{2.36}$$

Therefore, using estimates (2.12) and (2.13) of lemma 3,

$$\|\mathbf{H}_1^* - \mathbf{H}_2^*\|_{W(0,T)} \le T\left(A_R^* \frac{e^{\beta_R T} - 1}{\beta_R} + B_R^*\right)\|\mathbf{H}_1 - \mathbf{H}_2\|_{W(0,T)} \tag{2.37}$$

with $A_R^* = B_R + (1+\sqrt{\varepsilon_0\mu_0})\,C_R$ and $B_R^* = \gamma R + (1+\sqrt{\varepsilon_0\mu_0})\,D_R$.

The function $T \mapsto \Psi_R(T) = T\left(A_R^* \dfrac{e^{\beta_R T} - 1}{\beta_R} + B_R^*\right)$ is continuous, increasing and satisfies:

$$\Psi_R(0) = 0 \quad , \quad \lim_{t\to+\infty} \Psi_R(T) = +\infty \tag{2.38}$$

Defining $T_R^{(2)}$ as the unique solution of $\Psi_R(T) = \dfrac{1}{2}$, we have :

$$\forall T \le T_R^{(2)} \qquad \|\Phi(H_1) - \Phi(H_2)\|_{W(0,T)} \le \frac{1}{2}\|\mathbf{H}_1 - \mathbf{H}_2\|_{W(0,T)} \tag{2.39}$$

□

Remark 10 *$T_R^{(1)}$ and $T_R^{(2)}$ of lemmas 2 and 3 only depend on ε_0, μ_0, α, γ, K, $\mathbf{H}_s$ and R.*

Proof of theorem 3
Let us set $T_R = \min\left(T_R^{(1)}, T_R^{(2)}\right)$. According to lemmas 2 and 3, for $T = T_R$, Φ is a contraction from $E_R(T)$ into itself. Using the fixed point theorem, we deduce that Φ admits a unique fixed point $\mathbf{H}$ in $E_R(T)$ which means, according to lemma 1, that $(\mathbf{E},\mathbf{H},\mathbf{M}) = (\mathbf{E} = \mathcal{F} \circ \mathcal{M}(\mathbf{H})\,,\, \mathbf{H}\,,\, \mathbf{M} = \mathcal{M}(\mathbf{H}))$ is a strong solution of $(\mathcal{P})$. This also gives the uniqueness of the strong solution in the set $E_R(T)$. To conclude to the general uniqueness result, it suffices to remark that if $(\mathbf{E},\mathbf{H},\mathbf{M})$ is a strong solution on $[0,T]$ with $T \leq T_R^{(1)}$ then $\mathbf{H}$ necessarily belongs to $E_R(T)$ (cf. lemma 2). □

A direct consequence of theorem 3 is the uniqueness of any local solution.

Corollary 1 *Any strong solution of $(\mathcal{P})$ on $[0,T]$ is unique.*

Proof. Let $(\mathbf{E}_1,\mathbf{H}_1,\mathbf{M}_1)$ and $(\mathbf{E}_2,\mathbf{H}_2,\mathbf{M}_2)$ two solutions on $[0,T]$ with $T > T_R$. (If $T \leq T_R$ then theorem 1 applies.) Then they are in particular solutions on $[0,T_R]$. By theorem 1, they coincide on this interval $[0,T_R]$. And this argument can be repeated as long as necessary by shifting the initial time $t=0$ to $t=T_R$, and so on... □

2.2 Construction of the maximal solution

Let $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{M}_0)$ in V, we construct the maximal solution $(\mathbf{E},\mathbf{H},\mathbf{M})$ of $(\mathcal{P})$, as for ordinary differential equations, by considering :

$$T_{max} = \sup\{T > 0 \,/\, (\mathcal{P}) \text{ admits a strong solution } (\mathbf{E}_T,\mathbf{H}_T,\mathbf{M}_T) \text{ on } [0,T]\}$$

$$(\,T_{max} \in \mathbb{R}_+ \cup \{+\infty\}\,)$$

and define the maximal solution :

$$\left\{\begin{array}{lcl} (\mathbf{E},\mathbf{H}) & \in & C^0\left([0,T_{max}); H(\mathrm{curl}\,)\right) \cap C^1\left([0,T_{max}); (L^2)^3\right) \\ \mathbf{M} & \in & C^1\left([0,T_{max}); (L^2 \cap L^\infty)^3\right) \cap C^2\left([0,T_{max}); (L^2)^3\right) \end{array}\right.$$

by

$$\forall T < T_{max} \qquad (\mathbf{E},\mathbf{H},\mathbf{M})_{|[0,T]} = (\mathbf{E}_T,\mathbf{H}_T,\mathbf{M}_T)$$

(This definition is nos ambiguous thanks to the uniqueness result of corollary 1.) Note that the existence theorem 3 says that :

$$\|(\mathbf{E}_0,\mathbf{H}_0,\mathbf{M}_0)\| \leq R \quad \Rightarrow \quad T_{max} \geq T_R$$

The main result of this subsection is :

Theorem 6 *Let $(\mathbf{E},\mathbf{H},\mathbf{M})$ be the maximal solution of $(\mathcal{P})$ associated to initial data $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{M}_0) \in V$, one has the alternative :*

(i) $T_{max} = +\infty$ *: the solution is global*

(ii) $T_{max} < +\infty$ *and* $\lim\limits_{t \nearrow +\infty} \|(\mathbf{E}(\cdot,t),\mathbf{H}(\cdot,t),\mathbf{M}(\cdot,t))\|_V = +\infty$

Proof. It is very classical and we present it only for completeness. Assume that $T_{max} < +\infty$ and that

$$\|(\mathbf{E}(\cdot,t_n),\mathbf{H}(\cdot,t_n),\mathbf{M}(\cdot,t_n))\|_V \le R$$

for some $R>0$ and for some sequence $t_n \nearrow T_{max}$ (this would contradict point (ii)). Taking $t=t_n$ as the initial time, by applying theorem 1, we can define a strong solution of $(\mathcal{P})$ with initial data $(\mathbf{E}(\cdot,t),\mathbf{H}(\cdot,t),\mathbf{M}(\cdot,t))$ on the interval $[t_n, t_n+T_R]$. Therefore we obtain, by gluing, a strong solution on $[0, t_n+T_R]$. As $t_n \nearrow T_{max}$, we get $t_n+T_R > T_{max}$ which contradicts the definition of T_{max}. The theorem is then proved by contradiction. □

2.3 Proof of the global character of the solution

According to theorem 6, it suffices to obtain a priori estimates which shall prove that the quantity

$$\|(\mathbf{E}(\cdot,t),\mathbf{H}(\cdot,t),\mathbf{M}(\cdot,t))\|_V$$

(see (1.10)) can not blow up in finite time.

2.3.1 Step 1. Estimates on E, H and M

We have already seen (cf. proof of lemma 2) that

$$\text{a.e. } x \in \mathbb{R}\,,\ \forall t \ge 0 \qquad\qquad |\mathbf{M}(x,t)| \;=\; \mathbf{M}_0(x)|$$

which proves in particular that

$$\|\mathbf{M}(\cdot,t)\|_{L^q} \;=\; \|\mathbf{M}_0\|_{L^q} \qquad\qquad q=2,+\infty \tag{2.40}$$

Then we easily get a uniform estimate on $H_x(\cdot,t)$. Indeed, from remark 1, we deduce that :

$$H_x(x,t)+M_x(x,t) = \mathbf{H}_{0,x}(x)+\mathbf{M}_{0,x}(x)$$

Therefore :

$$\|H_x(\cdot,t)\|_{L^\infty} \le \|\mathbf{H}_{0,x}\|_{L^\infty} + 2\,\|\mathbf{M}_0\|_{L^\infty} \tag{2.41}$$

Now from Maxwell's equations we get :

$$\left\{\begin{aligned} \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\cdot\mathbf{E} - \text{curl }\mathbf{H}\cdot\mathbf{E} &\;=\; 0\\ \mu_0\frac{\partial \mathbf{H}}{\partial t}\cdot\mathbf{H} + \text{curl }\mathbf{E}\cdot\mathbf{H} &\;=\; -\mu_0\frac{\partial \mathbf{M}}{\partial t}\cdot\mathbf{H}\end{aligned}\right. \tag{2.42}$$

Summing these two equalities and integrating over $\mathbb{R}$ leads, after integrating by parts, to the following identity :

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}}\left(\varepsilon_0|\mathbf{E}|^2+\mu_0|\mathbf{H}|^2\right)\,dx\right\} \;=\; -\mu_0\int_{\mathbb{R}}\frac{\partial \mathbf{M}}{\partial t}\cdot\mathbf{H}\,dx \tag{2.43}$$

We now use the Landau-Lifschitz-Gilbert equation :

$$\frac{\partial \mathbf{M}}{\partial t} \;=\; \gamma\mathbf{H}_T\times\mathbf{M} + \frac{\alpha}{|\mathbf{M}|}\mathbf{M}\times\frac{\partial \mathbf{M}}{\partial t} \tag{2.44}$$

Taking the scalar product with $\mathbf{H}_T$, and using the notation $(\cdot,\cdot,\cdot)$ for the mixed product, we get

$$\frac{\partial \mathbf{M}}{\partial t}\cdot\mathbf{H}_T = \frac{\alpha}{|\mathbf{M}|}\left(\mathbf{M},\frac{\partial \mathbf{M}}{\partial t},\mathbf{H}_T\right)$$

Taking the scalar product of the Landau-Lifschitz-Gilbert equation by $\dfrac{\partial \mathbf{M}}{\partial t}$ gives :

$$\left|\frac{\partial \mathbf{M}}{\partial t}\right|^2 = \gamma\left(\frac{\partial \mathbf{M}}{\partial t},\mathbf{H}_T,\mathbf{M}\right)$$

Therefore, eliminating the mixed product :

$$\frac{\partial \mathbf{M}}{\partial t} \cdot \mathbf{H}_T = \frac{\alpha}{\gamma |\mathbf{M}|} \left| \frac{\partial \mathbf{M}}{\partial t} \right|^2 \tag{2.45}$$

(The meaning of the right hand side of this expression has to be understood in the sense we made precise in the remark 4.)

Now, by definition of $\mathbf{H}_T$, we have

$$\frac{\partial \mathbf{M}}{\partial t} \cdot \mathbf{H}_T = \frac{\partial \mathbf{M}}{\partial t} \cdot \mathbf{H} - KP(\mathbf{M}) \cdot P\left(\frac{\partial \mathbf{M}}{\partial t}\right) + \mathbf{H}_s \cdot \frac{\partial \mathbf{M}}{\partial t}$$

Using the fact that $|\mathbf{M}(x,t)| = |\mathbf{M}_0(x)|$, we note that :

$$\left\{ \begin{array}{l} \mathbf{H}_s \cdot \dfrac{\partial \mathbf{M}}{\partial t} = -\dfrac{1}{2}\dfrac{\partial}{\partial t}|\mathbf{H}_s - \mathbf{M}|^2 \\ -KP(\mathbf{M}) \cdot P\left(\dfrac{\partial \mathbf{M}}{\partial t}\right) = -\dfrac{1}{2}\dfrac{\partial}{\partial t}\left[K|P(\mathbf{M})|^2\right] \end{array} \right. \tag{2.46}$$

Therefore, we have :

$$\frac{\partial \mathbf{M}}{\partial t} \cdot \mathbf{H}_T = \frac{\partial \mathbf{M}}{\partial t} \cdot \mathbf{H} - \frac{1}{2}\frac{\partial}{\partial t}\left[|\mathbf{H}_s - \mathbf{M}|^2 + K|P(\mathbf{M})|^2\right] \tag{2.47}$$

that is to say, using (2.45) :

$$-\frac{\partial \mathbf{M}}{\partial t} \cdot \mathbf{H} = -\frac{\alpha}{\gamma |\mathbf{M}|} \left| \frac{\partial \mathbf{M}}{\partial t} \right|^2 - \frac{1}{2}\frac{\partial}{\partial t}\left[|\mathbf{H}_s - \mathbf{M}|^2 + K|P(\mathbf{M})|^2\right] \tag{2.48}$$

Plugging (2.48) into (2.43) leads to the energy identity :

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbf{R}} \left(\varepsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2 + \mu_0|\mathbf{H}_s - \mathbf{M}|^2 + \mu_0 K|P(\mathbf{M})|^2\right) dx\right\} + \frac{1}{\gamma}\int_{\mathbf{R}} \frac{\alpha}{|\mathbf{M}|}\left|\frac{\partial \mathbf{M}}{\partial t}\right|^2 dx \ = \ 0 \tag{2.49}$$

Remark 11 *Note that (2.40) and (2.49) are nothing but the two results* (i) *and* (ii) *of theorem 1.*

Setting

$$\mathcal{E}(\mathbf{E},\mathbf{H},\mathbf{M})(t) \ = \ \frac{1}{2}\int_{\mathbf{R}} \left(\varepsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2 + \mu_0|\mathbf{H}_s - \mathbf{M}|^2 + \mu_0 K|P(\mathbf{M})|^2\right) dx$$

equation (2.49) expresses that $\mathcal{E}(\mathbf{E},\mathbf{H},\mathbf{M})(t)$ is a decreasing function of time. This means in particular that :

$$\|\mathbf{E}(\cdot,t)\|_{L^2} \ , \ \|\mathbf{H}(\cdot,t)\|_{L^2}$$

remain bounded with respect to time, with an upper bound independent of t and related to the initial energy $\mathcal{E}(\mathbf{E}_0,\mathbf{H}_0,\mathbf{M}_0)$.

2.3.2 Step 2. Estimates on curl E and curl H

In order to get a control on curl $\mathbf{E}$ and curl $\mathbf{H}$, we establish in fact estimates on $\dfrac{\partial \mathbf{E}}{\partial t}$, $\dfrac{\partial \mathbf{H}}{\partial t}$ to finally conclude using Maxwell's equations.

First, let us remark that because of

$$\left|\frac{\partial \mathbf{M}}{\partial t}\right|^2 = \frac{\gamma^2}{1+\alpha^2}|\mathbf{H}_T \times \mathbf{M}|^2$$

we get (see the proof of lemma 2)

$$\left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} \leq \gamma \,\|\mathbf{M}_0\|_{L^\infty} \left(\|\mathbf{H}_T(\cdot,t)\|_{L^2} + \|K\|_{L^\infty} \|\mathbf{M}_0\|_{L^2} + \|\mathbf{H}(\cdot,t)\|_{L^2}\right) \tag{2.50}$$

Taking into account the previous estimate on $\|\mathbf{H}(\cdot,t)\|_{L^2}$, this proves that $\left\|\dfrac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2}$ is bounded uniformly in time. Now, to get L^2-estimates on $\dfrac{\partial \mathbf{E}}{\partial t}$ and $\dfrac{\partial \mathbf{H}}{\partial t}$, let us make some formal computations that may be justified (see remark 12). Let us differentiate Maxwell's equations with respect to time (which is not rigorous since we have not enough regularity) :

$$\left\{\begin{array}{rcl} \varepsilon_0 \dfrac{\partial^2 \mathbf{E}}{\partial t^2} - \operatorname{curl} \dfrac{\partial \mathbf{H}}{\partial t} & = & 0 \\ \mu_0 \dfrac{\partial^2 \mathbf{H}}{\partial t^2} + \operatorname{curl} \dfrac{\partial \mathbf{E}}{\partial t} & = & -\mu_0 \dfrac{\partial^2 \mathbf{M}}{\partial t^2} \end{array}\right. \tag{2.51}$$

Multiplying these two equalities respectively by $\dfrac{\partial \mathbf{E}}{\partial t}$ and $\dfrac{\partial \mathbf{H}}{\partial t}$ leads, after summation and integration in space, to the following equality :

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}}\left(\varepsilon_0\left|\frac{\partial \mathbf{E}}{\partial t}\right|^2 + \mu_0\left|\frac{\partial \mathbf{H}}{\partial t}\right|^2\right)dx\right\} = -\mu_0\int_{\mathbb{R}}\frac{\partial^2 \mathbf{M}}{\partial t^2}\cdot\frac{\partial \mathbf{H}}{\partial t}\,dx \tag{2.52}$$

(which is nothing but (2.43) written for the time derivatives of **E**, **H** and **M**.) After time integration between 0 and t, we get the following identity :

$$\begin{array}{rcl} \dfrac{1}{2}\displaystyle\int_{\mathbb{R}}\left(\varepsilon_0\left|\frac{\partial \mathbf{E}}{\partial t}\right|^2 + \mu_0\left|\frac{\partial \mathbf{H}}{\partial t}\right|^2\right)dx & = & \dfrac{1}{2}\displaystyle\int_{\mathbb{R}}\left(\frac{1}{\varepsilon_0}\left|\operatorname{curl}\mathbf{H}_0\right|^2 + \mu_0\left|\frac{\partial \mathbf{H}}{\partial t}(x,0)\right|^2\right)dx \\ & & -\displaystyle\int_0^t\int_{\mathbb{R}}\mu_0\frac{\partial^2 \mathbf{M}}{\partial t^2}\cdot\frac{\partial \mathbf{H}}{\partial t}\,dx\,ds \end{array} \tag{2.53}$$

Note that in (2.53), the quantity $\displaystyle\int_{\mathbb{R}}\left|\frac{\partial \mathbf{H}}{\partial t}(x,0)\right|^2 dx$ can be estimated by (using Maxwell's equations at time $t=0$) :

$$2\int_{\mathbb{R}}\left|\frac{\partial \mathbf{M}}{\partial t}(x,0)\right|^2 dx \;+\; \frac{2}{\mu_0^2}\int_{\mathbb{R}}\left|\operatorname{curl}\mathbf{E}_0\right|^2 dx$$

that is to say, using equality (2.17), by

$$2\gamma^2\int_{\mathbb{R}}\frac{1}{1+\alpha^2}\left|\mathbf{M}_0\times\mathbf{H}_0\right|^2 dx \;+\; \frac{2}{\mu_0^2}\int_{\mathbb{R}}\left|\operatorname{curl}\mathbf{E}_0\right|^2 dx$$

which is a constant which only depends on the data of the problem.

From now on, in the remaining of this proof, and only for reasons of simplicity, the letter C will represent a positive constant whose value may change from one line to another, but which will always have the property to depend only on the data of the problem (but not to depend on time).

First let us recall that (cf. (2.24) in the proof of lemma 2) :

$$\sqrt{1+\alpha^2}\left|\frac{\partial^2 \mathbf{M}}{\partial t^2}\right| \le \gamma\left(\left|\mathbf{H}_T\right|\left|\frac{\partial \mathbf{M}}{\partial t}\right| + \left|\frac{\partial \mathbf{H}_T}{\partial t}\right|\left|\mathbf{M}\right|\right)$$

from which we deduce

$$\left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}(\cdot,t)\right\|_{L^2} \le \gamma\left(\left\|\mathbf{H}_T(\cdot,t)\right\|_{L^\infty}\left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} + \left\|\frac{\partial \mathbf{H}_T}{\partial t}(\cdot,t)\right\|_{L^2}\left\|\mathbf{M}(\cdot,t)\right\|_{L^\infty}\right) \tag{2.54}$$

Estimate (2.25) on $\|\mathbf{H}_T(\cdot,t)\|_{L^\infty}$ can be written :

$$\|\mathbf{H}_T(\cdot,t)\|_{L^\infty} \le C\left(1+\|\mathbf{H}(\cdot,t)\|_{L^\infty}\right) \tag{2.55}$$

But

$$\begin{array}{rcl} \|\mathbf{H}(\cdot,t)\|_{L^\infty} & \le & \|H_x(\cdot,t)\|_{L^\infty} + \|H_y(\cdot,t)\|_{L^\infty} + \|H_z(\cdot,t)\|_{L^\infty} \\ & \le & \|H_x(\cdot,t)\|_{L^\infty} + \|H_y(\cdot,t)\|_{L^2} + \|H_z(\cdot,t)\|_{L^2} + \|\operatorname{curl}\mathbf{H}(\cdot,t)\|_{L^2} \end{array}$$

that is to say, taking (2.41) into account as well as (2.49), and using curl $\mathbf{H} = \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$:

$$\|\mathbf{H}(\cdot,t)\|_{L^\infty} \leq C\left(1 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}\right) \tag{2.56}$$

which also yields (cf. (2.55)) to

$$\|\mathbf{H}_T(\cdot,t)\|_{L^\infty} \leq C\left(1 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}\right) \tag{2.57}$$

On the other hand, equation (2.20) can be written

$$\left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} \leq C \tag{2.58}$$

Therefore

$$\|\mathbf{H}_T(\cdot,t)\|_{L^\infty} \left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} \leq C\left(1 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}\right) \tag{2.59}$$

In the same way, from $\dfrac{\partial \mathbf{H}_T}{\partial t} = \dfrac{\partial \mathbf{H}}{\partial t} - KP\left(\dfrac{\partial \mathbf{M}}{\partial t}\right)$ and using (2.58), we get :

$$\left\|\frac{\partial \mathbf{H}_T}{\partial t}(\cdot,t)\right\|_{L^2} \leq C\left(1 + \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2}\right) \tag{2.60}$$

Therefore

$$\left\|\frac{\partial \mathbf{H}_T}{\partial t}(\cdot,t)\right\|_{L^2} \|\mathbf{M}(\cdot,t)\|_{L^\infty} \leq C\left(1 + \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2}\right) \tag{2.61}$$

Thus (2.54) leads to (cf. (2.59) and (2.60)) :

$$\left\|\frac{\partial^2 \mathbf{M}}{\partial t^2}(\cdot,t)\right\|_{L^2} \leq C\left(1 + \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2} + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}\right) \tag{2.62}$$

Combining (2.53) and (2.62) gives :

$$F(t) \leq C\left(1 + t + \int_0^t F(s)ds\right) \quad \text{where } F(t) = \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2} + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2} \tag{2.63}$$

By Gronwall's lemma, we get :

$$\left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2} + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2} \leq Ce^{Ct} + \left(e^{Ct} - 1\right) \tag{2.64}$$

Using Maxwell's equations we get :

$$\left\{\begin{array}{lll} \|\text{curl } \mathbf{E}(\cdot,t)\|_{L^2} & \leq & \mu_0 \left\|\dfrac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2} + \mu_0 \left\|\dfrac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} \\ \|\text{curl } \mathbf{H}(\cdot,t)\|_{L^2} & \leq & \varepsilon_0 \left\|\dfrac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2} \end{array}\right. \tag{2.65}$$

which, using (2.64), provides the following estimate

$$\|\text{curl } \mathbf{E}(\cdot,t)\|_{L^2} + \|\text{curl } \mathbf{H}(\cdot,t)\|_{L^2} \leq C\left(1 + e^{Ct}\right) + e^{Ct} - 1 \tag{2.66}$$

Recapitulating our results, we see that estimates (2.40), (2.41), (2.49) and (2.66) prevent the V-norm of the solution $(\mathbf{E}(\cdot,t), \mathbf{H}(\cdot,t), \mathbf{M}(\cdot,t))$ from blowing up in finite time, which concludes our proof.

Remark 12 *Equation (2.53) can be rigorously justified by replacing time derivatives by finite differences. Let us introduce, if $u \in C^0([0,T_{max}); X)$, the difference operator D_h :*

$$\forall T < T^*,\ \forall h \in [0, T_{max} - T] \qquad D_h u(t) = \frac{u(t+h) - u(t)}{h}$$

Note that $D_h u \in C^0\left(0,T;X\right)$. Applying D_h to Maxwell's equations (here $X = \left(L^2(\mathbb{R})\right)^3$), we get :

$$\left\{\begin{array}{lcl} \varepsilon_0 \dfrac{\partial}{\partial t}\left(D_h\mathbf{E}\right) - curl\ \left(D_h\mathbf{H}\right) & = & 0 \\ \mu_0 \dfrac{\partial}{\partial t}\left(D_h\mathbf{H}\right) + curl\ \left(D_h\mathbf{E}\right) & = & -\mu_0 \dfrac{\partial}{\partial t}\left(D_h\mathbf{M}\right) \end{array}\right. \tag{2.67}$$

Multiplying the first equation of (2.67) by $D_h\mathbf{E}$ and the second one by $D_h\mathbf{H}$ leads,after summation, space integration and time integration, to the equality

$$\begin{array}{lcl} \dfrac{1}{2}\displaystyle\int_{\mathbf{R}}\left(\varepsilon_0\left|D_h\mathbf{E}(x,t)\right|^2 + \mu_0\left|D_h\mathbf{H}(x,t)\right|^2\right)dx & = & \dfrac{1}{2}\displaystyle\int_{\mathbf{R}}\left(\varepsilon_0\left|D_h\mathbf{E}(x,0)\right|^2 + \mu_0\left|D_h\mathbf{H}(x,0)\right|^2\right)dx \\ & & -\displaystyle\int_0^t\int_{\mathbf{R}} D_h\frac{\partial\mathbf{M}}{\partial t}(x,s)D_h\mathbf{H}(x,s)\,dxds \end{array} \tag{2.68}$$

As $(\mathbf{E},\mathbf{H}) \in C^1\left([0,T_{max});(L^2)^3\right)^2$, we have the following convergences, which are uniform with respect to time if $t \in [0,T]$, $T < T_{max}$:

$$\left\{\begin{array}{lcl} \displaystyle\lim_{h\to 0}\|D_h\mathbf{E}(t)\|_{L^2}^2 & = & \left\|\dfrac{\partial\mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}^2 \\ \displaystyle\lim_{h\to 0}\|D_h\mathbf{H}(t)\|_{L^2}^2 & = & \left\|\dfrac{\partial\mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2}^2 \end{array}\right. \tag{2.69}$$

In the same way, as $\mathbf{M} \in C^2\left([0,T_{max});(L^2)^3\right)$, we have

$$\lim_{h\to 0}\int_0^t\int_{\mathbf{R}} D_h\frac{\partial\mathbf{M}}{\partial t}(x,s)D_h\mathbf{H}(x,s)\,dxds = \int_0^t\int_{\mathbf{R}}\frac{\partial^2\mathbf{M}}{\partial t^2}\cdot\frac{\partial\mathbf{H}}{\partial t}\,dxds \tag{2.70}$$

Then one obtains (2.53) by taking the limit of (2.68) when h tends to 0.

3 Proof of theorem 2

In subsection 2.2 we obtain estimates on $\|\text{curl }\mathbf{E}(\cdot,t)\|_{L^2}$ and $\|\text{curl }\mathbf{H}(\cdot,t)\|_{L^2}$ which allow an exponential growth of these quantities for large time. This is essentially due to the fact that we simply made a brute estimate of the right hand side in (2.53). This can be improved as follows. As we have :

$$\frac{\partial^2\mathbf{M}}{\partial t^2} = \gamma\left(\mathbf{H}_T\times\frac{\partial\mathbf{M}}{\partial t} + \frac{\partial\mathbf{H}_T}{\partial t}\times\mathbf{M}\right) + \frac{\alpha}{|\mathbf{M}|}\mathbf{M}\times\frac{\partial^2\mathbf{M}}{\partial t^2}$$

we easily get, after scalar products by $\dfrac{\partial\mathbf{H}_T}{\partial t}$ and $\dfrac{\partial^2\mathbf{M}}{\partial t^2}$ respectively :

$$\left\{\begin{array}{lcl} \dfrac{\partial^2\mathbf{M}}{\partial t^2}\cdot\dfrac{\partial\mathbf{H}_T}{\partial t} & = & \gamma\left(\mathbf{H}_T,\dfrac{\partial\mathbf{M}}{\partial t},\dfrac{\partial\mathbf{H}_T}{\partial t}\right) + \dfrac{\alpha}{|\mathbf{M}|}\left(\mathbf{M},\dfrac{\partial^2\mathbf{M}}{\partial t^2},\dfrac{\partial\mathbf{H}_T}{\partial t}\right) \\ \left|\dfrac{\partial^2\mathbf{M}}{\partial t^2}\right|^2 & = & \gamma\left(\dfrac{\partial\mathbf{H}_T}{\partial t},\dfrac{\partial\mathbf{M}}{\partial t},\dfrac{\partial^2\mathbf{M}}{\partial t^2}\right) + \gamma\left(\mathbf{H}_T,\dfrac{\partial\mathbf{M}}{\partial t},\dfrac{\partial^2\mathbf{M}}{\partial t^2}\right) \end{array}\right. \tag{3.1}$$

We eliminate $\left(\mathbf{M},\dfrac{\partial^2\mathbf{M}}{\partial t^2},\dfrac{\partial\mathbf{H}_T}{\partial t}\right)$ and we obtain :

$$\gamma\frac{\partial^2\mathbf{M}}{\partial t^2}\cdot\frac{\partial\mathbf{H}_T}{\partial t} - \frac{\alpha}{|\mathbf{M}|}\left|\frac{\partial^2\mathbf{M}}{\partial t^2}\right|^2 = \gamma^2\left(\mathbf{H}_T,\frac{\partial\mathbf{M}}{\partial t},\frac{\partial\mathbf{H}_T}{\partial t}\right) - \gamma\frac{\alpha}{|\mathbf{M}|}\left(\mathbf{H}_T,\frac{\partial\mathbf{M}}{\partial t},\frac{\partial^2\mathbf{M}}{\partial t^2}\right) \tag{3.2}$$

Using $\dfrac{\partial\mathbf{H}_T}{\partial t} = \dfrac{\partial\mathbf{H}}{\partial t} - KP\left(\dfrac{\partial\mathbf{M}}{\partial t}\right)$, we get :

$$\begin{array}{lcl} \dfrac{\partial^2\mathbf{M}}{\partial t^2}\cdot\dfrac{\partial\mathbf{H}}{\partial t} & = & \dfrac{1}{\gamma}\dfrac{\alpha}{|\mathbf{M}|}\left|\dfrac{\partial^2\mathbf{M}}{\partial t^2}\right|^2 - \dfrac{1}{2}\dfrac{\partial}{\partial t}\left[K|P(\dfrac{\partial\mathbf{M}}{\partial t})|^2\right] \\ & & +\gamma\left(\mathbf{H}_T,\dfrac{\partial\mathbf{M}}{\partial t},\dfrac{\partial\mathbf{H}_T}{\partial t}\right) - \dfrac{\alpha}{|\mathbf{M}|}\left(\mathbf{H}_T,\dfrac{\partial\mathbf{M}}{\partial t},\dfrac{\partial^2\mathbf{M}}{\partial t^2}\right) \end{array} \tag{3.3}$$

that we plug into (2.53) to finally obtain :

$$\Phi(t)+\frac{1}{\gamma}\int_0^t\int_{\mathbb{R}}\frac{\alpha}{|\mathbf{M}|}\left|\frac{\partial^2\mathbf{M}}{\partial t^2}\right|^2 dxds \le \Phi(0)+S_1(t)+S_2(t) \tag{3.4}$$

where we have set :

$$\begin{aligned}
\Phi(t) &= \frac{1}{2}\int_{\mathbb{R}}\left(\varepsilon_0\left|\frac{\partial\mathbf{E}}{\partial t}\right|^2+\mu_0\left|\frac{\partial\mathbf{H}}{\partial t}\right|^2+\mu_0K\left|P\left(\frac{\partial\mathbf{M}}{\partial t}\right)\right|^2\right)dx\\
S_1(t) &= \int_0^t\int_{\mathbb{R}}\frac{\alpha}{|\mathbf{M}|}\left(\mathbf{H}_T,\frac{\partial\mathbf{M}}{\partial t},\frac{\partial^2\mathbf{M}}{\partial t^2}\right)dxds\\
S_2(t) &= -\gamma\int_0^t\int_{\mathbb{R}}\left(\mathbf{H}_T,\frac{\partial\mathbf{M}}{\partial t},\frac{\partial\mathbf{H}_T}{\partial t}\right)dxds
\end{aligned}$$

Since $\left|\frac{\partial\mathbf{M}}{\partial t}\right| \le \frac{\gamma}{\sqrt{1+\alpha^2}}|\mathbf{M}||\mathbf{H}_T|$, it is easy to see that (remember that the letter C will represent a positive constant whose value may change from one line to another)

$$|S_1(t)+S_2(t)|\le C\int_0^t\int_{\mathbb{R}}|\mathbf{H}_T|^2\left(\left|\frac{\partial^2\mathbf{M}}{\partial t^2}\right|+\left|\frac{\partial\mathbf{H}_T}{\partial t}\right|\right)dxds$$

From (2.24) we know that :

$$\begin{aligned}
\left|\frac{\partial^2\mathbf{M}}{\partial t^2}\right| &\le C\left(|\mathbf{H}_T|\left|\frac{\partial\mathbf{M}}{\partial t}\right|+\left|\frac{\partial\mathbf{H}_T}{\partial t}\right||\mathbf{M}|\right)\\
&\le C\left(|\mathbf{H}_T|^2+\left|\frac{\partial\mathbf{H}_T}{\partial t}\right|\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
|S_1(t)+S_2(t)| &\le C\int_0^t\int_{\mathbb{R}}\left(|\mathbf{H}_T|^4+|\mathbf{H}_T|^2\left|\frac{\partial\mathbf{H}_T}{\partial t}\right|\right)dxds\\
&\le C\int_0^t\left(\|\mathbf{H}_T(\cdot,s)\|_{L^4}^4+\|\mathbf{H}_T(\cdot,s)\|_{L^4}^2\left\|\frac{\partial\mathbf{H}_T}{\partial t}(\cdot,s)\right\|_{L^2}\right)ds
\end{aligned} \tag{3.5}$$

Note that, as $\left|\frac{\partial\mathbf{H}_T}{\partial t}\right|\le C\left(\left|\frac{\partial\mathbf{H}}{\partial t}\right|+\left|\frac{\partial\mathbf{M}}{\partial t}\right|\right)\le C\left(\left|\frac{\partial\mathbf{H}}{\partial t}\right|+|\mathbf{H}_T|\right)$ we have

$$\begin{aligned}
\left\|\frac{\partial\mathbf{H}_T}{\partial t}(\cdot,t)\right\|_{L^2} &\le C\left(\left\|\frac{\partial\mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2}+\|\mathbf{H}_T(\cdot,t)\|_{L^2}\right)\\
&\le C\left(1+\left\|\frac{\partial\mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2}\right)
\end{aligned}$$

this yields :

$$|S_1(t)+S_2(t)|\le C\int_0^t\left(\|\mathbf{H}_T(\cdot,s)\|_{L^4}^4+\|\mathbf{H}_T(\cdot,s)\|_{L^4}^2+\|\mathbf{H}_T(\cdot,s)\|_{L^4}^2\left\|\frac{\partial\mathbf{H}}{\partial t}(\cdot,s)\right\|_{L^2}\right)dxds \tag{3.6}$$

From the equality $\mathbf{H}_T=\mathbf{H}+\mathbf{H}_s+KP(\mathbf{M})$, as we have $L^\infty\cap L^2$ uniform estimates on $\mathbf{M}$, we get, since $L^\infty\cap L^2\hookrightarrow L^4$ (and since $\mathbf{H}_s\in L^4$) :

$$\|\mathbf{H}_T(\cdot,t)\|_{L^4}\le C\,(1+\|\mathbf{H}(\cdot,t)\|_{L^4})$$

which yields

$$|S_1(t)+S_2(t)|\le C\int_0^t\left(1+\|\mathbf{H}(\cdot,s)\|_{L^4}^4+\|\mathbf{H}(\cdot,s)\|_{L^4}^2+\left\|\frac{\partial\mathbf{H}}{\partial t}(\cdot,s)\right\|_{L^2}+\|\mathbf{H}(\cdot,s)\|_{L^4}^2\left\|\frac{\partial\mathbf{H}}{\partial t}(\cdot,s)\right\|_{L^2}\right)dxdsdxds \tag{3.7}$$

Now, setting $\mathbf{H}_\tau = (0, H_y, H_z)^t$, we have

$$\begin{aligned} \|\mathbf{H}(\cdot,t)\|_{L^4} &\leq \|H_x(\cdot,t)\|_{L^4} + \|\mathbf{H}_\tau(\cdot,t)\|_{L^4} \\ &\leq C\left(1 + \|\mathbf{H}_\tau(\cdot,t)\|_{L^4}\right) \end{aligned}$$

(since we have a uniform $L^2 \cap L^\infty$ estimate on H_x). By Gagliardo-Niremberg's interpolation inequality :

$$\begin{aligned} \|\mathbf{H}_\tau(\cdot,t)\|_{L^4} &\leq \|\text{curl } \mathbf{H}(\cdot,t)\|_{L^2}^{\frac{1}{4}} \|\mathbf{H}_\tau(\cdot,t)\|_{L^2}^{\frac{3}{4}} \\ &\leq C\left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}^{\frac{1}{4}} \end{aligned} \tag{3.8}$$

Then

$$\left\{ \begin{aligned} \|\mathbf{H}_\tau(\cdot,t)\|_{L^4}^2 &\leq C\left(1 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}^{\frac{1}{2}}\right) \\ \|\mathbf{H}_\tau(\cdot,t)\|_{L^4}^2 &\leq C\left(1 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}\right) \end{aligned} \right. \tag{3.9}$$

This gives

$$\begin{aligned} S_1(t) + S_2(t) \leq\ & C\int_0^t \left(1 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,s)\right\|_{L^2}^{\frac{1}{2}} + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,s)\right\|_{L^2}\right) ds \\ & + C\int_0^t \left(1 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,s)\right\|_{L^2}^{\frac{1}{2}}\right) \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,s)\right\|_{L^2} ds \end{aligned} \tag{3.10}$$

Therefore, if we set $F(t) = \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2}^2 + \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2}^2$, we have

$$|S_1(t) + S_2(t)| \leq C\int_0^t \left(1 + F(s)^{\frac{3}{4}}\right) ds$$

This means that, from (3.4), we get :

$$F(t) \leq Ct + \int_0^t F(s)^{\frac{3}{4}} ds \tag{3.11}$$

By a Gronwall's type lemma, this yields

$$F(t) \leq C\left(1 + t^4\right) \tag{3.12}$$

Which that we have a polynomial estimate for the L^2 norms of $\frac{\partial \mathbf{E}}{\partial t}$ and $\frac{\partial \mathbf{H}}{\partial t}$:

$$\left\{ \begin{aligned} \left\|\frac{\partial \mathbf{E}}{\partial t}(\cdot,t)\right\|_{L^2} &\leq C\left(1 + t^2\right) \\ \left\|\frac{\partial \mathbf{H}}{\partial t}(\cdot,t)\right\|_{L^2} &\leq C\left(1 + t^2\right) \end{aligned} \right. \tag{3.13}$$

and thus, using Maxwell's equations :

$$\left\{ \begin{aligned} \|\text{curl } \mathbf{E}(\cdot,t)\|_{L^2} &\leq C\left(1 + t^2\right) \\ \|\text{curl } \mathbf{H}(\cdot,t)\|_{L^2} &\leq C\left(1 + t^2\right) \end{aligned} \right. \tag{3.14}$$

4 Proof of the intermediate results

In this last part we give the proof of some intermediate results that have been used in section 2 – namely theorem (4), lemma (3) and theorem (5). These proofs complete the paper.

4.1 Proof of theorem 4

Let us first note that, $\mathbf{H}$ being given in $W(0,T)$, the evolution equation :

$$\left\{\begin{aligned} \frac{\partial \mathbf{M}}{\partial t} &= \gamma \mathbf{H}_T \times \mathbf{M} + \frac{\alpha}{|\mathbf{M}|}\mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} \\ \mathbf{H}_T &= \mathbf{H} + \mathbf{H}_s - KP(\mathbf{M}) \end{aligned}\right. \tag{4.1}$$

can be rewritten as :

$$\left\{\begin{aligned} \frac{\partial \mathbf{M}}{\partial t} &= \frac{\gamma}{1+\alpha^2}\Big(\mathbf{H}_T \times \mathbf{M} + \frac{\alpha}{|\mathbf{M}|}\mathbf{M} \times (\mathbf{H}_T \times \mathbf{M})\Big) \\ \mathbf{H}_T &= \mathbf{H} + \mathbf{H}_s - KP(\mathbf{M}) \end{aligned}\right. \tag{4.2}$$

that we are going to consider as an abstract evolution equation in the Banach space :

$$E = \left(L^2 \cap L^\infty\right)^3 \tag{4.3}$$

equipped with the norm :

$$\|\mathbf{M}\|_E = \|\mathbf{M}\|_{L^2} + \|\mathbf{M}\|_{L^\infty} \tag{4.4}$$

Let us introduce the map :

$$F(\mathbf{M},t) \ : \ E \times [0,T] \longrightarrow E \tag{4.5}$$

defined by,$\forall \mathbf{M} \in E, \forall t \in [0,T]$,

$$F(\mathbf{M},t) = \frac{\gamma}{1+\alpha^2}\Big(\mathbf{H}_T(\mathbf{M},t) \times \mathbf{M} + \frac{\alpha}{|\mathbf{M}|}\mathbf{M} \times (\mathbf{H}_T(\mathbf{M},t) \times \mathbf{M})\Big) \tag{4.6}$$

where $\mathbf{H}_T(\mathbf{M},t) = \mathbf{H}(\cdot,t) + \mathbf{H}_s - KP(\mathbf{M})$. Note that as $W(0,T) \subset C^0\left(0,T;(L^2 \cap L^\infty)^3\right)$, $F(\mathbf{M},t)$ belongs to E thanks to the fact that $\left(L^2 \cap L^\infty\right)^3$ is an algebra. The main property of $F(\mathbf{M},t)$ is the following :

Lemma 5 *For any $R > 0$, there exists a constant $C_R(T)$ such that, if $B_R = \{\mathbf{M} \in E \,/\, \|\mathbf{M}\|_E \leq R\}$:*

$$\forall\, (\mathbf{M}_1, \mathbf{M}_2) \in B_R^2, \forall t \leq T, \quad \|F(\mathbf{M}_1,t) - F(\mathbf{M}_2,t)\|_E \leq C_R \|\mathbf{M}_1 - \mathbf{M}_2\|_E$$

As a direct consequence of this lemma, we have, by Cauchy-Lipschitz theorem :

Corollary 2 *For any $\mathbf{M}_0 \in E$, the problem :*

$$\left\{\begin{aligned} &\frac{\partial \mathbf{M}}{\partial t} = F(\mathbf{M},t) \quad t \in [0,T] \\ &\mathbf{M}(t=0) = \mathbf{M}_0 \end{aligned}\right.$$

has a unique solution $\mathbf{M} \in C^1(0,T;E)$.

Proof of lemma 5. Obviously it suffices to get estimates of

$$\|\mathbf{H}_T(\mathbf{M}_1,t) \times \mathbf{M}_1\|_E - \|\mathbf{H}_T(\mathbf{M}_2,t) \times \mathbf{M}_2\|_E$$

and

$$\left\|\frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times (\mathbf{H}_T(\mathbf{M}_1,t) \times \mathbf{M}_1) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times (\mathbf{H}_T(\mathbf{M}_2,t) \times \mathbf{M}_2)\right\|_E$$

4.1.1 Step 1. Estimate on $\|\mathbf{H}_T(\mathbf{M}_1,t)\times\mathbf{M}_1\|_E - \|\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_2\|_E$

We have

$$\mathbf{H}_T(\mathbf{M}_1,t)\times\mathbf{M}_1-\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_2=\mathbf{H}_T(\mathbf{M}_1,t)\times(\mathbf{M}_1-\mathbf{M}_2)+(\mathbf{H}_T(\mathbf{M}_1,t)-\mathbf{H}_T(\mathbf{M}_2,t))\times\mathbf{M}_2$$

Remark that we have the property

$$\forall(u,v)\in E^2 \qquad \|u\times v\|_E\le\|u\|_E\,\|v\|_E \tag{4.7}$$

This yields

$$\begin{aligned}\|\mathbf{H}_T(\mathbf{M}_1,t)\times\mathbf{M}_1-\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_2\|_E &\le \|\mathbf{H}_T(\mathbf{M}_1,t)\|_E\,\|\mathbf{M}_1-\mathbf{M}_2\|_E\\ &+\|\mathbf{M}_2\|_E\,\|\mathbf{H}_T(\mathbf{M}_1,t)-\mathbf{H}_T(\mathbf{M}_2,t)\|_E\end{aligned}$$

Since $\mathbf{M}_1\in B_R$, we have :

$$\|\mathbf{H}_T(\mathbf{M}_1,t)\|_E\le C_T+\|K\|_{L^\infty}R \quad \text{where } C_T=\|\mathbf{H}\|_{W(0,T)}+\|\mathbf{H}_s\|_{L^\infty}$$

Moreover, it is obvious that

$$\|\mathbf{H}_T(\mathbf{M}_1,t)-\mathbf{H}_T(\mathbf{M}_2,t)\|_E\le\|K\|_{L^\infty}\|\mathbf{M}_1-\mathbf{M}_2\|_E$$

Therefore

$$\|\mathbf{H}_T(\mathbf{M}_1,t)\times\mathbf{M}_1-\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_2\|_E\le(C_T+2\|K\|_{L^\infty}R)\,\|\mathbf{M}_1-\mathbf{M}_2\|_E \tag{4.8}$$

4.1.2 Step 2. Estimate on $\left\|\frac{\mathbf{M}_1}{|\mathbf{M}_1|}\times(\mathbf{H}_T(\mathbf{M}_1,t)\times\mathbf{M}_1)-\frac{\mathbf{M}_2}{|\mathbf{M}_2|}\times(\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_2)\right\|_E$

Let us set

$$D = \frac{\mathbf{M}_1}{|\mathbf{M}_1|}\times(\mathbf{H}_T(\mathbf{M}_1,t)\times\mathbf{M}_1)-\frac{\mathbf{M}_2}{|\mathbf{M}_2|}\times(\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_2)$$

We can write

$$D = D_1+D_2 \tag{4.9}$$

where we have set :

$$\left\{\begin{aligned} D_1 &= \frac{\mathbf{M}_1}{|\mathbf{M}_1|}\times[(\mathbf{H}_T(\mathbf{M}_1,t)-\mathbf{H}_T(\mathbf{M}_2,t))\times\mathbf{M}_1]\\ D_2 &= \frac{\mathbf{M}_1}{|\mathbf{M}_1|}\times(\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_1)-\frac{\mathbf{M}_2}{|\mathbf{M}_2|}\times(\mathbf{H}_T(\mathbf{M}_2,t)\times\mathbf{M}_2)\end{aligned}\right.$$

Let us note that :

$$\begin{aligned}\|D_1\|_E &\le \|\mathbf{M}_1\|_E\,\|\mathbf{H}_T(\mathbf{M}_1,t)-\mathbf{H}_T(\mathbf{M}_2,t)\|_E\\ &\le R\,\|K\|_{L^\infty}\|\mathbf{M}_1-\mathbf{M}_2\|_E\end{aligned} \tag{4.10}$$

Now we remark that, setting $\mathbf{H}_2=\mathbf{H}_T(\mathbf{M}_2,t)$,

$$D_2=(|\mathbf{M}_1|-|\mathbf{M}_2|)\,\mathbf{H}_2-\left[\frac{(\mathbf{M}_1\cdot\mathbf{H}_2)\mathbf{M}_1}{|\mathbf{M}_1|}-\frac{(\mathbf{M}_2\cdot\mathbf{H}_2)\mathbf{M}_2}{|\mathbf{M}_2|}\right] \tag{4.11}$$

As $||\mathbf{M}_1|-|\mathbf{M}_2||\le|\mathbf{M}_1-\mathbf{M}_2|$, we see that :

$$\begin{aligned}\|(|\mathbf{M}_1|-|\mathbf{M}_2|)\,\mathbf{H}_2\|_E &\le \|\mathbf{M}_1-\mathbf{M}_2\|_E\,\|\mathbf{H}_2\|_E\\ &\le (C_T+R\,\|K\|_{L^\infty})\,\|\mathbf{M}_1-\mathbf{M}_2\|_E\end{aligned} \tag{4.12}$$

Now let us consider :

$$D_3 = \frac{(\mathbf{M}_1 \cdot \mathbf{H}_2)\mathbf{M}_1}{|\mathbf{M}_1|} - \frac{(\mathbf{M}_2 \cdot \mathbf{H}_2)\mathbf{M}_2}{|\mathbf{M}_2|}$$

The j^{th} component D_3^j of D_3 is given by :

$$D_3^j = \sum_{k=1}^{3} \left(\frac{\mathbf{M}_1^j \mathbf{M}_1^k}{|\mathbf{M}_1|} - \frac{\mathbf{M}_2^j \mathbf{M}_2^k}{|\mathbf{M}_2|} \right) \mathbf{H}_2^k$$

Note that the functions $\phi_{jk} : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$\phi_{jk}(\mathbf{x}) \ = \ \frac{\mathbf{x}^j \mathbf{x}^k}{|\mathbf{x}|}$$

are Lipschitz continuous :

$$\phi_{jk}(\mathbf{x}) - \phi_{jk}(\mathbf{y}) \leq C^* |\mathbf{x} - \mathbf{y}| \tag{4.13}$$

Now :

$$\|D_3\|_E = \sum_j \left\| D_3^j \right\|_E \leq \sum_{k=1}^{3} \left\| \mathbf{H}_2^k \right\|_E \|\phi_{jk}(\mathbf{M}_1) - \phi_{jk}(\mathbf{M}_2)\|_E$$

From (4.13), we deduce that :

$$\|\phi_{jk}(\mathbf{M}_1) - \phi_{jk}(\mathbf{M}_2)\|_E \leq C^* \|\mathbf{M}_1 - \mathbf{M}_2\|_E \tag{4.14}$$

Therefore

$$\begin{aligned} \|D_3\|_E &\leq C^* \|\mathbf{H}_2\|_E \|\mathbf{M}_1 - \mathbf{M}_2\|_E \\ &\leq C^* (C_T + R \|K\|_{L^\infty}) \|\mathbf{M}_1 - \mathbf{M}_2\|_E \end{aligned} \tag{4.15}$$

From (4.12) and (4.15), we deduce (using (4.11)) :

$$\|D_2\|_E \leq (1 + C^*) (C_T + R \|K\|_{L^\infty}) \|\mathbf{M}_1 - \mathbf{M}_2\|_E \tag{4.16}$$

and from (4.10) and (4.16), we get, by (4.9) :

$$\|D\|_E \leq (2 + C^*) (C_T + R \|K\|_{L^\infty}) \|\mathbf{M}_1 - \mathbf{M}_2\|_E \tag{4.17}$$

It is then immediate to conclude from (4.8) and (4.17).

To conclude the proof of the existence part of the theorem, it remains to prove that

$$\mathbf{M} \in C^2 \left(0, T; (L^2)^3\right)$$

Because of equation (4.2), it suffices to note that :

$$\left. \begin{array}{r} \mathbf{H}_T \in C^1 \left(0, T; (L^2)^3\right) \\ \\ \mathbf{M} \in C^1 \left(0, T; (L^\infty)^3\right) \end{array} \right\} \implies \mathbf{H}_T \times \mathbf{M} \in C^1 \left(0, T; (L^2)^3\right) \tag{4.18}$$

(Think to the formula $\dfrac{\partial}{\partial t}(\mathbf{H}_T \times \mathbf{M}) = \dfrac{\partial \mathbf{H}_T}{\partial t} \times \mathbf{M} + \mathbf{H}_T \times \dfrac{\partial \mathbf{M}}{\partial t}$ and to the fact that the product of a function in $C^1\left(0, T; (L^2)^3\right)$ by a function in $C^1\left(0, T; (L^\infty)^3\right)$ belongs to $C^1\left(0, T; (L^2)^3\right)$.) In the same way

$$\left. \begin{array}{r} \mathbf{H}_T \in C^1 \left(0, T; (L^2)^3\right) \\ \\ \mathbf{M} \in C^1 \left(0, T; (L^\infty)^3\right) \\ \\ |\mathbf{M}(x,t)| = |\mathbf{M}_0(x)| \end{array} \right\} \implies \frac{\mathbf{M}}{|\mathbf{M}|} \times (\mathbf{H}_T \times \mathbf{M}) \in C^1 \left(0, T; (L^2)^3\right) \tag{4.19}$$

Equations (4.18) and (4.19) imply, since $\alpha \in L^\infty$, that :

$$\frac{\partial \mathbf{M}}{\partial t} \in C^1\left(0,T;(L^2)^3\right) \tag{4.20}$$

what we wanted to show. □

Remark 13 *To see that* $\mathbf{H}_T = \mathbf{H} + \mathbf{H}_s - KP(\mathbf{M}) \in C^1\left(0,T;(L^2)^3\right)$ *(cf. (4.18)), it suffices to note that :*

$$\begin{cases} \mathbf{M} \in C^1(0,T;E) \quad \Rightarrow \quad KP(\mathbf{M}) \in C^1\left(0,T;(L^2)^3\right) \\[2ex] \mathbf{H} \in W(0,T) \quad \Rightarrow \quad \mathbf{H} \in C^1\left(0,T;(L^2)^3\right) \end{cases}$$

Note that a priori $\mathbf{H}$ *does not belong to* $C^1\left(0,T;(L^\infty)^3\right)$ *– only to* $C^0(0,T;L^\infty)$ *– which explains why we cannot conclude that* $\mathbf{M} \in C^2(0,T;E)$.

Remark 14 *To prove (4.13), it suffices to remark that* ϕ_{jk} *is* C^1 *in* $\mathbb{R} \setminus \{0\}$ *and that*

$$\frac{\partial \phi_{jk}}{\partial x_l}(\mathbf{x}) = \delta_{jl}\frac{x_k}{|x|} + \delta_{kl}\frac{x_j}{|x|} - \frac{x_j x_k x_l}{|x|^3}$$

This proves that, $\forall x \neq 0$, $\left|\frac{\partial \phi_{jk}}{\partial x_l}(\mathbf{x})\right| \leq 3$ *and thus that* $|\nabla \phi_{jk}(\mathbf{x})| \leq 3\sqrt{3}$. *Therefore, if* $\mathbf{x}$ *and* $\mathbf{y}$ *are such that* $0 \notin [\mathbf{x},\mathbf{y}]$, *by Rolle's theorem, we get*

$$|\nabla \phi_{jk}(\mathbf{x}) - \nabla \phi_{jk}(\mathbf{y})| \leq 3\sqrt{3}|\mathbf{x} - \mathbf{y}|$$

If $0 \in [\mathbf{x},\mathbf{y}]$, *which means that* $\exists \theta, \mathbf{x} = \lambda\theta, \mathbf{y} = -\mu\theta$ *with* $|\theta| = 1$, $\lambda > 0$ *and* $\mu > 0$, *we have :*

$$\phi_{jk}(\mathbf{x}) = |\lambda|\phi_{jk}(\theta) \qquad\qquad \phi_{jk}(\mathbf{y}) = |\mu|\phi_{jk}(\theta)$$

then

$$|\phi_{jk}(\mathbf{x}) - \phi_{jk}(\mathbf{y})| = |\phi_{jk}(\theta)|\,||\lambda| - |\mu|| \leq |\lambda| + |\mu| = |\mathbf{x} - \mathbf{y}|$$

Therefore we have proved (4.13) with $C^* = 3\sqrt{3}$.

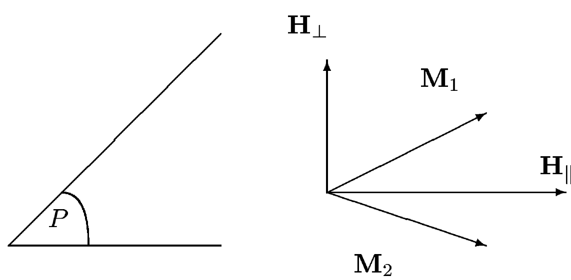
Remark 15 *It is also possible to get a control of the term* D_2 *of (4.9) – and thus to prove (4.13) – using geomatrical arguments. More exactly we have the following identity :*

$$\forall(\mathbf{M}_1,\mathbf{M}_2,\mathbf{H}) \in \left(\mathbb{R}^3\right)^3, \quad \Delta(\mathbf{M}_1,\mathbf{M}_2,\mathbf{H}) \;=\; \left|\frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times (\mathbf{H} \times \mathbf{M}_1) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times (\mathbf{H} \times \mathbf{M}_2)\right| \;\leq\; |\mathbf{H}|\,|\mathbf{M}_1 - \mathbf{M}_2| \tag{4.21}$$

that we prove now. We decompose

$$\mathbf{H} \;=\; \mathbf{H}_{||} \;+\; \mathbf{H}_\perp$$

where $\mathbf{H}_{||}$ *is in the plane* Π *defined by* $\mathbf{M}_1$ *and* $\mathbf{M}_2$, *while* $\mathbf{H}_\perp$ *is perpendicular to this plane. (Of course we assume* $\frac{\mathbf{M}_1}{|\mathbf{M}_1|} \neq \frac{\mathbf{M}_2}{|\mathbf{M}_2|}$, *otherwise the result is obvious.)*



Note that $|\mathbf{H}|^2 = |\mathbf{H}_{\|}|^2 + |\mathbf{H}_{\perp}|^2$, *and that*

$$\mathbf{M}_i \times (\mathbf{H}_{\|} \times \mathbf{M}_i) \in \Pi$$

and

$$\mathbf{M}_i \times (\mathbf{H}_{\perp} \times \mathbf{M}_i) \perp \Pi$$

Thus we have with Pythagore's identity

$$\begin{aligned}
|\Delta(\mathbf{M}_1,\mathbf{M}_2,\mathbf{H})|^2 &= \left| \frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_1\right) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_2\right) \right. \\
&\qquad \left. + \frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_1) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_2) \right|^2 \\
&= \left| \frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_1\right) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_2\right) \right|^2 \\
&\qquad + \left| \frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_1) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_2) \right|^2
\end{aligned}$$

As $\mathbf{H}_{\perp}$ *is perpendicular to* $\mathbf{M}_1$ *and* $\mathbf{M}_2$, *we have*

$$\frac{\mathbf{M}_i}{|\mathbf{M}_i|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_i) = |\mathbf{M}_i|\,\mathbf{H}_{\perp}$$

which allows us to say that

$$\begin{aligned}
\left| \frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_1) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_2) \right|^2 &= |\,(|\mathbf{M}_1| - |\mathbf{M}_2|)\;\mathbf{H}_{\perp}\,|^2 \\
&= |\mathbf{H}_{\perp}|^2\;(|\mathbf{M}_1| - |\mathbf{M}_2|)^2
\end{aligned}$$

and to conclude that

$$\left| \frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_1) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times (\mathbf{H}_{\perp} \times \mathbf{M}_2) \right|^2 \leq |\mathbf{H}_{\perp}|^2\,|\mathbf{M}_1 - \mathbf{M}_2|^2 \tag{4.22}$$

On the other hand, concerning $\mathbf{H}_{\|}$, *we have*

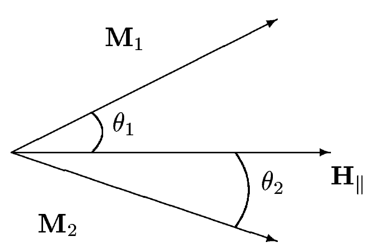
$$\frac{\mathbf{H}_{\|} \times \mathbf{M}_i}{|\mathbf{H}_{\|}|\,|\mathbf{M}_i|\,\sin\theta_i} \times \left[\frac{\mathbf{M}_i}{|\mathbf{M}_i|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_i\right)\right] = \sin\theta_i\,|\mathbf{H}_{\|}|\,\mathbf{M}_i$$

where

$$\theta_1 = \left(\widehat{\mathbf{H}_{\|},\mathbf{M}_1}\right) \neq 0$$

$$\theta_2 = \left(\widehat{\mathbf{H}_{\|},\mathbf{M}_2}\right) \neq 0$$

$$\theta = (\widehat{\mathbf{M}_1,\mathbf{M}_2}) = \theta_2 - \theta_1$$



Since

$$\frac{\mathbf{H}_{\|} \times \mathbf{M}_1}{|\mathbf{H}_{\|}|\,|\mathbf{M}_1|\,\sin\theta_1} = \frac{\mathbf{H}_{\|} \times \mathbf{M}_2}{|\mathbf{H}_{\|}|\,|\mathbf{M}_2|\,\sin\theta_2} = \mathbf{u}$$

the crossproduct by this vector $\mathbf{u}$ *defines a rotation – that is to say an isometry – which maps* $\dfrac{\mathbf{M}_i}{|\mathbf{M}_i|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_i\right)$ *on* $\sin\theta_i\,|\mathbf{H}_{\|}|\,\mathbf{M}_i$, *and thus we have*

$$\left| \frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_1\right) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_2\right) \right|^2 = |\mathbf{H}_{\|}|^2\,|\,\sin\theta_1\,\mathbf{M}_1 - \sin\theta_2\,\mathbf{M}_2\,|^2 \tag{4.23}$$

It is then straightforward to compute

$$\begin{aligned}
|\,\mathbf{M}_1 - \mathbf{M}_2\,|^2 - |\,\sin\theta_1\,\mathbf{M}_1 - \sin\theta_2\,\mathbf{M}_2\,|^2 &= |\mathbf{M}_1|^2 + |\mathbf{M}_2|^2 - 2\cos\theta|\mathbf{M}_1||\mathbf{M}_2| \\
&\quad - \left(\sin^2\theta_1|\mathbf{M}_1|^2 + \sin^2\theta_2|\mathbf{M}_2|^2 - 2\sin\theta_1\sin\theta_2|\mathbf{M}_1||\mathbf{M}_2|\right) \\
&= \cos^2\theta_1|\mathbf{M}_1|^2 + \cos^2\theta_2|\mathbf{M}_2|^2 - 2\cos\theta_1\cos\theta_2|\mathbf{M}_1||\mathbf{M}_2| \\
&= \left(\cos\theta_1|\mathbf{M}_1| - \cos\theta_2|\mathbf{M}_2|\right)^2 \geq 0
\end{aligned} \tag{4.24}$$

which shows, with (4.23), that

$$\left|\frac{\mathbf{M}_1}{|\mathbf{M}_1|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_1\right) - \frac{\mathbf{M}_2}{|\mathbf{M}_2|} \times \left(\mathbf{H}_{\|} \times \mathbf{M}_2\right)\right|^2 \leq |\mathbf{H}_{\|}|^2\,|\,\mathbf{M}_1 - \mathbf{M}_2\,|^2 \tag{4.25}$$

Taking into account that $|\mathbf{H}|^2 = |\mathbf{H}_{\|}|^2 + |\mathbf{H}_{\perp}|^2$, *we get the result by summing (4.22) and (4.25).*

4.2 Proof of lemma 3

Before establishing (2.12) and (2.13), we first show in step 1 that

$$\|\mathbf{M}_1 - \mathbf{M}_2\|_{L^\infty} \leq A\,\frac{e^{-\beta_R t} - 1}{\beta_R}\,\|\mathbf{H}_1 - \mathbf{H}_2\|_{L^\infty(0,T;L^\infty)}$$

with $A = \gamma\,\|\alpha\|_{L^\infty}$. Then we repeat the same arguments in steps 2 and 3, but for the derivatives of the fields, to obtain the two inequalities of lemma 3, and also similar results in L^2 norm.

In what follows, we denote by $\mathbf{M}_1$ and $\mathbf{M}_2$ the respective solutions of (4.1) with $\mathbf{H} = \mathbf{H}_1$ and $\mathbf{H} = \mathbf{H}_2$, and denote by $\mathbf{H}_T^1$ and $\mathbf{H}_T^2$ the corresponding values of $\mathbf{H}_T$. Each step consists in finding suitable upper bounds to the derivative of the norm of $(\mathbf{M}_1 - \mathbf{M}_2)$ (or their derivatives) and to conclude with Gronwall's lemma.

4.2.1 Step 1.

Let us first remark that, since $|\mathbf{M}_1| = |\mathbf{M}_2| = |\mathbf{M}_0|$:

$$\begin{aligned}
\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} &= \frac{\gamma}{1+\alpha^2}\left[\mathbf{H}_T^1 \times (\mathbf{M}_1 - \mathbf{M}_2) + \mathbf{M}_2 \times \left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right)\right] \\
&\quad + \frac{\alpha\gamma}{|\mathbf{M}_0|(1+\alpha^2)}\mathbf{M}_2 \times \left[\mathbf{H}_T^1 \times (\mathbf{M}_1 - \mathbf{M}_2) + \left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right) \times \mathbf{M}_2\right] \\
&\quad + \frac{\alpha\gamma}{|\mathbf{M}_0|(1+\alpha^2)}(\mathbf{M}_1 - \mathbf{M}_2) \times \mathbf{H}_T^1 \times \mathbf{M}_1
\end{aligned}$$

Consequently, after scalar product by $(\mathbf{M}_1 - \mathbf{M}_2)$:

$$\frac{1}{2}\frac{d}{dt}\,|\mathbf{M}_1 - \mathbf{M}_2|^2 = F \tag{4.26}$$

where we have set :

$$\begin{aligned}
F &= \frac{\gamma}{1+\alpha^2}\left(\mathbf{M}_2\,,\,\mathbf{H}_T^1 - \mathbf{H}_T^2\,,\,\mathbf{M}_1 - \mathbf{M}_2\right) \\
&\quad + \frac{\alpha\gamma}{|\mathbf{M}_0|(1+\alpha^2)}\left(\mathbf{M}_2\,,\,\mathbf{H}_T^1 \times (\mathbf{M}_1 - \mathbf{M}_2)\,,\,\mathbf{M}_1 - \mathbf{M}_2\right) \\
&\quad + \frac{\alpha\gamma}{|\mathbf{M}_0|(1+\alpha^2)}\left(\mathbf{M}_2\,,\,\left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right) \times \mathbf{M}_2\,,\,\mathbf{M}_1 - \mathbf{M}_2\right)
\end{aligned} \tag{4.27}$$

Note that

$$\begin{aligned}
\|F(\cdot,t)\|_{L^\infty} &\leq \gamma\,\|\mathbf{M}_0\|_{L^\infty}\left\|\left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right)(\cdot,t)\right\|_{L^\infty}\|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty} \\
&\quad + \gamma\,\|\alpha\|_{L^\infty}\left\|\mathbf{H}_T^1(\cdot,t)\right\|_{L^\infty}\|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty} \\
&\quad + \gamma\,\|\alpha\|_{L^\infty}\left\|\left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right)(\cdot,t)\right\|_{L^\infty}\|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty}
\end{aligned} \tag{4.28}$$

It is immediate to see that

$$\left\| \left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right)(\cdot,t)\right\|_{L^\infty} \leq \|K\|_{L^\infty} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty} + \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty} \tag{4.29}$$

Therefore

$$\begin{aligned} \|F(\cdot,t)\|_{L^\infty} \quad \leq \quad & \left[\gamma \|K\|_{L^\infty} (\|\mathbf{M}_0\|_{L^\infty} + \|\alpha\|_{L^\infty}) + \gamma \|\alpha\|_{L^\infty} \left\|\mathbf{H}_T^1(\cdot,t)\right\|_{L^\infty}\right] \\ & + \gamma \|\alpha\|_{L^\infty} \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty} \end{aligned} \tag{4.30}$$

Now, it suffices to remark that (for $j = 1, 2$) :

$$\left.\begin{array}{l} \mathbf{H}_T^j \;=\; \mathbf{H}_j + \mathbf{H}_s - KP(\mathbf{M}_j) \\[2ex] \|\mathbf{M}_j(\cdot,t)\|_{L^\infty} \leq R \\[2ex] \|\mathbf{H}_j\|_{W(0,T)} \leq 1 + \tau R + \sigma R^2 \end{array}\right\} \quad \Rightarrow \quad \left\|\mathbf{H}_T^1(\cdot,t)\right\|_{L^\infty} \leq C_1(R) \tag{4.31}$$

where $C_1(R)$ only depends on ε_0, μ_0, α, γ, K, $\mathbf{H}_s$ and $\mathbf{M}$. Therefore, setting

$$\beta_R = C_1(R)\gamma \|\alpha\|_{L^\infty} + \gamma \|K\|_{L^\infty} (\|\mathbf{M}_0\|_{L^\infty} + \|\alpha\|_{L^\infty})$$

we have

$$\|F(\cdot,t)\|_{L^\infty} \leq \beta_R \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty}^2 + \gamma \|\alpha\|_{L^\infty} \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty} \tag{4.32}$$

Setting $G(t) = \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty}^2$, from (4.26) and (4.32), we get :

$$G'(t) \leq 2\beta_R G(t) + 2\gamma \|\alpha\|_{L^\infty} \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty} G^{\frac{1}{2}}(t)$$

which we can write :

$$\frac{d}{dt}\left(e^{-2\beta_R t} G(t)\right) \leq 2\gamma \|\alpha\|_{L^\infty} \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty} \left(e^{-2\beta_R t} G(t)\right)^{\frac{1}{2}} e^{-\beta_R t}$$

from which we get, setting $A = \gamma \|\alpha\|_{L^\infty}$,

$$\|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty} \;\leq\; A\,\frac{e^{-\beta_R t} - 1}{\beta_R} \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty(0,T;L^\infty)} \tag{4.33}$$

Note that we obtain an analogous estimate for the L^2 norm. This will be useful for the step 3 of the present proof. Indeed, from (4.27), we easily deduce :

$$\begin{aligned} \|F(\cdot,t)\|_{L^1} \quad \leq \quad & \gamma \|\mathbf{M}_0\|_{L^\infty} \left\|\left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right)(\cdot,t)\right\|_{L^2} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2} \\ & + \gamma \|\alpha\|_{L^\infty} \left\|\mathbf{H}_T^1(\cdot,t)\right\|_{L^\infty} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2} \\ & + \gamma \|\alpha\|_{L^\infty} \left\|\left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right)(\cdot,t)\right\|_{L^2} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2} \end{aligned} \tag{4.34}$$

As $\left\|\left(\mathbf{H}_T^1 - \mathbf{H}_T^2\right)(\cdot,t)\right\|_{L^2} \leq \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^2} + \|K\|_{L^\infty} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2}$, it is easy to see that we get for $\|F(\cdot,t)\|_{L^1}$ the same estimate than the one obtained for $\|F(\cdot,t)\|_{L^\infty}$ in (4.30). It suffices to replace the norm $\|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^\infty}$ by $\|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2}$ and the norm $\|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty}$ by $\|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^2}$. Therefore

$$\|F(\cdot,t)\|_{L^1} \leq \beta_R \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2}^2 + \gamma \|\alpha\|_{L^\infty} \|(\mathbf{H}_1 - \mathbf{H}_2)(\cdot,t)\|_{L^\infty} \|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2}$$

Then, proceeding exactly as before, we get :

$$\|(\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t)\|_{L^2} \;\leq\; A\,\frac{e^{-\beta_R t} - 1}{\beta_R} \|\mathbf{H}_1 - \mathbf{H}_2\|_{L^\infty(0,T;L^2)} \tag{4.35}$$

4.2.2 Step 2.

Note that

$$\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} - \alpha \frac{\mathbf{M}_1}{|\mathbf{M}_0|} \times \left(\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right) = \gamma \left[\mathbf{H}_T^1 \times (\mathbf{M}_1 - \mathbf{M}_2) + \left(\mathbf{H}_T^1 - \mathbf{H}_T^2 \right) \times \mathbf{M}_2 \right] + \frac{\alpha}{|\mathbf{M}_0|} \frac{\partial \mathbf{M}_2}{\partial t} \times (\mathbf{M}_1 - \mathbf{M}_2)$$

from which we easily deduce :

$$\begin{aligned} \left| \frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right| &\leq \frac{\gamma}{\sqrt{1+\alpha^2}} \left[\left| \mathbf{H}_T^1 \right| |\mathbf{M}_1 - \mathbf{M}_2| + \left| \mathbf{H}_T^1 - \mathbf{H}_T^2 \right| |\mathbf{M}_2| \right] \\ &+ \frac{\alpha\gamma}{\sqrt{1+\alpha^2}} \frac{1}{|\mathbf{M}_0|} \left| \frac{\partial \mathbf{M}_2}{\partial t} \right| |\mathbf{M}_1 - \mathbf{M}_2| \end{aligned} \tag{4.36}$$

We know (cf. formula (2.17), subsection 2.1) that :

$$\frac{1}{|\mathbf{M}_0|} \left| \frac{\partial \mathbf{M}_2}{\partial t} \right| \leq \frac{\gamma}{\sqrt{1+\alpha^2}} \left| \mathbf{H}_T^2 \right|$$

Therefore

$$\begin{aligned} \left\| \left(\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right)(\cdot,t) \right\|_{L^\infty} &\leq \gamma \left\| \mathbf{H}_T^1(\cdot,t) \right\|_{L^\infty} \left\| (\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t) \right\|_{L^\infty} + \left\| \left(\mathbf{H}_T^1 - \mathbf{H}_T^2 \right)(\cdot,t) \right\|_{L^\infty} \left\| \mathbf{M}_2(\cdot,t) \right\|_{L^\infty} \\ &+ \gamma^2 \|\alpha\|_{L^\infty} \left\| \mathbf{H}_T^2(\cdot,t) \right\|_{L^\infty} \left\| (\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t) \right\|_{L^\infty} \end{aligned} \tag{4.37}$$

Using (4.29), we get

$$\begin{aligned} \left\| \left(\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right)(\cdot,t) \right\|_{L^\infty} &\leq \gamma \left\| \left(\mathbf{H}_T^1 - \mathbf{H}_T^2 \right)(\cdot,t) \right\|_{L^\infty} \|\mathbf{M}_0\|_{L^\infty} \\ &+ \gamma \left[\left\| \mathbf{H}_T^1(\cdot,t) \right\|_{L^\infty} + \gamma \|\alpha\|_{L^\infty} \left\| \mathbf{H}_T^2(\cdot,t) \right\|_{L^\infty} \|K\|_{L^\infty} \|\mathbf{M}_0\|_{L^\infty} \right] \left\| (\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t) \right\|_{L^\infty} \end{aligned} \tag{4.38}$$

that we can rewrite, thanks to (4.31) :

$$\left\| \left(\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right)(\cdot,t) \right\|_{L^\infty} \leq C_2(R) \left\| (\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t) \right\|_{L^\infty} + \gamma R \left\| \left(\mathbf{H}_T^1 - \mathbf{H}_T^2 \right)(\cdot,t) \right\|_{L^\infty} \tag{4.39}$$

where $C_2(R) = \gamma \left[(1 + \gamma \|\alpha\|_{L^\infty}) + \|K\|_{L^\infty} R \right]$. Using estimate (4.33), we get with $B_R = A\, C_2(R)$:

$$\left\| \left(\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right)(\cdot,t) \right\|_{L^\infty} \leq \left[B_R \frac{e^{-\beta_R t} - 1}{\beta_R} + \gamma R \right] \left\| \mathbf{H}_T^1 - \mathbf{H}_T^2 \right\|_{W(0,T)} \tag{4.40}$$

since $\left\| \left(\mathbf{H}_T^1 - \mathbf{H}_T^2 \right)(\cdot,t) \right\|_{L^\infty} \leq \left\| \mathbf{H}_T^1 - \mathbf{H}_T^2 \right\|_{W(0,T)}$. Note that we also obtain an estimate in the L^2 norm. Indeed, from (4.36),we get

$$\left\| \left(\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right)(\cdot,t) \right\|_{L^2} \leq \gamma \left\| \mathbf{H}_T^1(\cdot,t) \right\|_{L^\infty} \left\| (\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t) \right\|_{L^2} + \left\| \left(\mathbf{H}_T^1 - \mathbf{H}_T^2 \right)(\cdot,t) \right\|_{L^2} \left\| \mathbf{M}_2(\cdot,t) \right\|_{L^\infty} \tag{4.41}$$

Then the same computations as before lead to

$$\left\| \left(\frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right)(\cdot,t) \right\|_{L^2} \leq C_2(R) \left\| (\mathbf{M}_1 - \mathbf{M}_2)(\cdot,t) \right\|_{L^2} + \gamma R \left\| \left(\mathbf{H}_T^1 - \mathbf{H}_T^2 \right)(\cdot,t) \right\|_{L^2} \tag{4.42}$$

which, combined with (4.35), leads to :

$$\left\| \frac{\partial \mathbf{M}_1}{\partial t} - \frac{\partial \mathbf{M}_2}{\partial t} \right\|_{L^\infty(0,T;L^2)} \leq T \left[B_R \frac{e^{-\beta_R t} - 1}{\beta_R} + \gamma R \right] \left\| \mathbf{H}_T^1 - \mathbf{H}_T^2 \right\|_{W(0,T)} \tag{4.43}$$

which is nothing but (2.12), the first of the two estimates we wanted to obtain.

4.2.3 Step 3.

One easily computes that :

$$\begin{aligned}\frac{\partial^2\mathbf{M}_1}{\partial t^2}-\frac{\partial^2\mathbf{M}_2}{\partial t^2}-\alpha\frac{\mathbf{M}_1}{|\mathbf{M}_0|}\times\left(\frac{\partial^2\mathbf{M}_1}{\partial t^2}-\frac{\partial^2\mathbf{M}_2}{\partial t^2}\right) &= \gamma\left[(\mathbf{H}_T^1-\mathbf{H}_T^2)\times\frac{\partial\mathbf{M}_1}{\partial t}+\mathbf{H}_T^2\times\left(\frac{\partial\mathbf{M}_1}{\partial t}-\frac{\partial\mathbf{M}_2}{\partial t}\right)\right]\\ &+ \gamma\left[\frac{\partial\mathbf{H}_T^2}{\partial t}\times(\mathbf{M}_1-\mathbf{M}_2)+\left(\frac{\partial\mathbf{H}_T^1}{\partial t}-\frac{\partial\mathbf{H}_T^2}{\partial t}\right)\times\mathbf{M}_1\right]\\ &+ \frac{\alpha}{|\mathbf{M}_0|}(\mathbf{M}_1-\mathbf{M}_2)\times\frac{\partial^2\mathbf{M}_2}{\partial t^2}\end{aligned}$$

which yields :

$$\begin{aligned}\left|\frac{\partial^2\mathbf{M}_1}{\partial t^2}-\frac{\partial^2\mathbf{M}_2}{\partial t^2}\right| \le\ & \frac{\gamma}{\sqrt{1+\alpha^2}}\left[\left|\mathbf{H}_T^2\right|\left|\frac{\partial\mathbf{M}_1}{\partial t}-\frac{\partial\mathbf{M}_2}{\partial t}\right|+\left|\mathbf{H}_T^1-\mathbf{H}_T^2\right|\left|\frac{\partial\mathbf{M}_1}{\partial t}\right|\right]\\ &+\frac{\gamma}{\sqrt{1+\alpha^2}}\left[|\mathbf{M}_1|\left|\frac{\partial\mathbf{H}_T^1}{\partial t}-\frac{\partial\mathbf{H}_T^2}{\partial t}\right|+|\mathbf{M}_1-\mathbf{M}_2|\left|\frac{\partial\mathbf{H}_T^2}{\partial t}\right|\right]\\ &+\frac{\alpha\gamma}{\sqrt{1+\alpha^2}}\frac{1}{|\mathbf{M}_0|}\left|\frac{\partial^2\mathbf{M}_2}{\partial t^2}\right||\mathbf{M}_1-\mathbf{M}_2|\end{aligned} \tag{4.44}$$

Using the estimates (cf. (2.17) and (2.24)) :

$$\left\{\begin{aligned}&\left|\frac{\partial\mathbf{M}_j}{\partial t}\right|\le\frac{\gamma}{\sqrt{1+\alpha^2}}|\mathbf{M}_0|\left|\mathbf{H}_T^j\right|\\ &\left|\frac{\partial^2\mathbf{M}_j}{\partial t^2}\right|\le\frac{\gamma}{\sqrt{1+\alpha^2}}|\mathbf{M}_0|\left|\frac{\partial\mathbf{H}_T^j}{\partial t}\right|+\frac{\gamma^2}{1+\alpha^2}|\mathbf{M}_0|\left|\mathbf{H}_T^j\right|\end{aligned}\right.$$

We obtain

$$\begin{aligned}&\left\|\left(\frac{\partial^2\mathbf{M}_1}{\partial t^2}-\frac{\partial^2\mathbf{M}_2}{\partial t^2}\right)(\cdot,t)\right\|_{L^2} \le\\ &\gamma\left[\gamma\|\mathbf{M}_0\|_{L^\infty}\left\|\mathbf{H}_T^1(\cdot,t)\right\|_{L^2}\left\|(\mathbf{H}_T^1-\mathbf{H}_T^2)(\cdot,t)\right\|_{L^2}+\left\|\mathbf{H}_T^2(\cdot,t)\right\|_{L^\infty}\left\|\left(\frac{\partial\mathbf{M}_1}{\partial t}-\frac{\partial\mathbf{M}_2}{\partial t}\right)(\cdot,t)\right\|_{L^2}\right]\\ &+\gamma\left[\|\mathbf{M}_0\|_{L^\infty}\left\|\left(\frac{\partial\mathbf{H}_T^1}{\partial t}-\frac{\partial\mathbf{H}_T^2}{\partial t}\right)(\cdot,t)\right\|_{L^2}+\left\|\frac{\partial\mathbf{H}_T^1}{\partial t}(\cdot,t)\right\|_{L^2}\|(\mathbf{M}_1-\mathbf{M}_2)(\cdot,t)\|_{L^\infty}\right]\\ &+\ \|\alpha\|_{L^\infty}\left[\gamma\|\mathbf{M}_0\|_{L^\infty}\left\|\frac{\partial\mathbf{H}_T^2}{\partial t}(\cdot,t)\right\|_{L^2}+\gamma^2\|\mathbf{M}_0\|_{L^2}\left\|\mathbf{H}_T^2(\cdot,t)\right\|_{L^\infty}\right]\|(\mathbf{M}_1-\mathbf{M}_2)(\cdot,t)\|_{L^\infty}\end{aligned} \tag{4.45}$$

From $\dfrac{\partial\mathbf{H}_T^j}{\partial t}=\dfrac{\partial\mathbf{H}^j}{\partial t}-KP\left(\dfrac{\partial\mathbf{M}_j}{\partial t}\right)$, we deduce

$$\left\{\begin{aligned}&\left\|\frac{\partial\mathbf{H}_T^j}{\partial t}(\cdot,t)\right\|_{L^2}\le\left\|\frac{\partial\mathbf{H}^j}{\partial t}(\cdot,t)\right\|_{L^2}+\|K\|_{L^\infty}\left\|\frac{\partial\mathbf{M}^j}{\partial t}(\cdot,t)\right\|_{L^2}\\ &\left\|\left(\frac{\partial\mathbf{H}_T^1}{\partial t}-\frac{\partial\mathbf{H}_T^2}{\partial t}\right)(\cdot,t)\right\|_{L^2}\le\left\|\left(\frac{\partial\mathbf{H}_1}{\partial t}-\frac{\partial\mathbf{H}_2}{\partial t}\right)(\cdot,t)\right\|_{L^2}+\|K\|_{L^\infty}\left\|\left(\frac{\partial\mathbf{M}_1}{\partial t}-\frac{\partial\mathbf{M}_2}{\partial t}\right)(\cdot,t)\right\|_{L^2}\end{aligned}\right.$$

which leads to

$$\begin{aligned}
&\left\|\left(\frac{\partial^2 \mathbf{M}_1}{\partial t^2}-\frac{\partial^2 \mathbf{M}_2}{\partial t^2}\right)(\cdot,t)\right\|_{L^2} \leq \gamma\left[\|\mathbf{M}_0\|_{L^\infty}\left\|\left(\frac{\partial \mathbf{H}_1}{\partial t}-\frac{\partial \mathbf{H}_2}{\partial t}\right)(\cdot,t)\right\|_{L^2}+\|K\|_{L^\infty}\left\|\left(\frac{\partial \mathbf{M}_1}{\partial t}-\frac{\partial \mathbf{M}_2}{\partial t}\right)(\cdot,t)\right\|_{L^2}\right] \\
&\quad +\gamma\left[\gamma\|\mathbf{M}_0\|_{L^\infty}\left\|\mathbf{H}_T^1(\cdot,t)\right\|_{L^2}\left(\|(\mathbf{H}_1-\mathbf{H}_2)(\cdot,t)\|_{L^2}+\|K\|_{L^\infty}\|(\mathbf{M}_1-\mathbf{M}_2)(\cdot,t)\|_{L^2}\right)\right. \\
&\qquad\qquad \left.+\left\|\mathbf{H}_T^2(\cdot,t)\right\|_{L^\infty}\left\|\left(\frac{\partial \mathbf{M}_1}{\partial t}-\frac{\partial \mathbf{M}_2}{\partial t}\right)(\cdot,t)\right\|_{L^2}\right] \\
&\quad +\|\alpha\|_{L^\infty}\left[\gamma\|\mathbf{M}_0\|_{L^\infty}\left(\left\|\frac{\partial \mathbf{H}_2}{\partial t}(\cdot,t)\right\|_{L^2}+\|K\|_{L^\infty}\left\|\frac{\partial \mathbf{M}_2}{\partial t}(\cdot,t)\right\|_{L^2}\right)\right. \\
&\qquad\qquad \left.+\gamma^2\|\mathbf{M}_0\|_{L^2}\left\|\mathbf{H}_T^2(\cdot,t)\right\|_{L^\infty}\right]\|(\mathbf{M}_1-\mathbf{M}_2)(\cdot,t)\|_{L^\infty}
\end{aligned} \tag{4.46}$$

Using the fact that $\mathbf{H}_j \in E_R(T)$ and the fact that $\|\mathbf{M}_0\|_{L^p} \leq R$, $p=2,+\infty$, it is easy to verify that there exists constants $\{C_l(R), l=3,4,5,6,7\}$ such that :

$$\begin{aligned}
\left\|\left(\frac{\partial^2 \mathbf{M}_1}{\partial t^2}-\frac{\partial^2 \mathbf{M}_2}{\partial t^2}\right)(\cdot,t)\right\|_{L^2} \leq\; & C_3(R)\|(\mathbf{H}_1-\mathbf{H}_2)(\cdot,t)\|_{L^2}+C_4(R)\left\|\left(\frac{\partial \mathbf{H}_T^1}{\partial t}-\frac{\partial \mathbf{H}_T^2}{\partial t}\right)(\cdot,t)\right\|_{L^2} \\
& +C_5(R)\|(\mathbf{M}_1-\mathbf{M}_2)(\cdot,t)\|_{L^2}+C_6(R)\|(\mathbf{M}_1-\mathbf{M}_2)(\cdot,t)\|_{L^\infty} \\
& +C_7(R)\left\|\left(\frac{\partial \mathbf{M}_1}{\partial t}-\frac{\partial \mathbf{M}_2}{\partial t}\right)(\cdot,t)\right\|_{L^\infty}
\end{aligned} \tag{4.47}$$

Then, using estimates (4.33), (4.35), (4.40) and (4.43), we can deduvce the existence of two constants C_R and D_R such that :

$$\left\|\frac{\partial^2 \mathbf{M}_1}{\partial t^2}-\frac{\partial^2 \mathbf{M}_2}{\partial t^2}\right\|_{L^\infty(0,T;L^2)} \leq T\left[C_R\frac{e^{-\beta_R t}-1}{\beta_R}+D_R\right]\left\|\mathbf{H}_T^1-\mathbf{H}_T^2\right\|_{W(0,T)} \tag{4.48}$$

which is nothing but (2.13).

4.3 Proof of theorem 5

The existence and uniqueness of $(\mathbf{E}, H^*)$ is a simple consequence of the properties of linear Maxwell's equations (it suffices for instance to apply Hille-Yosida's theorem). we now proceed to derive estimate (2.32). From :

$$\left\{\begin{aligned}
\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\cdot\mathbf{E}-\operatorname{curl}\mathbf{H}^*\cdot\mathbf{E} &= 0 \\
\mu_0\frac{\partial \mathbf{H}^*}{\partial t}\cdot\mathbf{H}^*+\operatorname{curl}\mathbf{E}\cdot\mathbf{H}^* &= -\mu_0\frac{\partial \mathbf{M}}{\partial t}\cdot\mathbf{H}^*
\end{aligned}\right. \tag{4.49}$$

We deduce after summation and integration in space :

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}}\left(\varepsilon_0|\mathbf{E}|^2+\mu_0|\mathbf{H}^*|^2\right)dx\right\} = -\mu_0\int_{\mathbb{R}}\frac{\partial \mathbf{M}}{\partial t}\cdot\mathbf{H}^*\,dx \tag{4.50}$$

Let us set $U(t)=\frac{1}{2}\int_{\mathbb{R}}\left(\varepsilon_0|\mathbf{E}|^2+\mu_0|\mathbf{H}^*|^2\right)dx$; we get

$$\begin{aligned}
\frac{dU}{dt} &\leq \sqrt{2\mu_0}\left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2} U^{\frac{1}{2}} \\
\Rightarrow\; U^{-\frac{1}{2}}\frac{dU}{dt} &\leq \sqrt{2\mu_0}\left\|\frac{\partial \mathbf{M}}{\partial t}(\cdot,t)\right\|_{L^2}
\end{aligned}$$

Therefore, after integration in time,

$$\forall t \leq T \qquad U^{\frac{1}{2}}(t) \leq U^{\frac{1}{2}}(0) + \sqrt{\frac{\mu_0}{2}} \left\| \frac{\partial \mathbf{M}}{\partial t}(\cdot, t) \right\|_{L^2}$$

Now using the fact that $\|\mathbf{H}^*\|_{L^2} \leq \sqrt{\frac{2}{\mu_0}} U^{\frac{1}{2}}(t)$, and the same way that $\|\mathbf{E}\|_{L^2} \leq \sqrt{\frac{2}{\varepsilon_0}} U^{\frac{1}{2}}(t)$, we get :

$$\|\mathbf{H}^*\|_{L^\infty(0,T;L^2)} \leq \sqrt{\frac{2}{\mu_0}} U^{\frac{1}{2}}(0) + \left\| \frac{\partial \mathbf{M}}{\partial t} \right\|_{L^1(0,T;L^2)} \tag{4.51}$$

$$\|\mathbf{E}\|_{L^\infty(0,T;L^2)} \leq \sqrt{\frac{2}{\varepsilon_0}} U^{\frac{1}{2}}(0) + \sqrt{\frac{\varepsilon_0}{\mu_0}} \left\| \frac{\partial \mathbf{M}}{\partial t} \right\|_{L^1(0,T;L^2)} \tag{4.52}$$

Applying the same estimates to the equations after integration in time we obtain – we omit the details of the justification which can be done following the method of remark 12.

$$\left\| \frac{\partial \mathbf{H}^*}{\partial t} \right\|_{L^\infty(0,T;L^2)} \leq \sqrt{\frac{2}{\mu_0}} \dot{U}^{\frac{1}{2}}(0) + \left\| \frac{\partial^2 \mathbf{M}}{\partial t^2} \right\|_{L^1(0,T;L^2)} \tag{4.53}$$

$$\left\| \frac{\partial \mathbf{E}}{\partial t} \right\|_{L^\infty(0,T;L^2)} \leq \sqrt{\frac{2}{\varepsilon_0}} \dot{U}^{\frac{1}{2}}(0) + \sqrt{\frac{\mu_0}{\varepsilon_0}} \left\| \frac{\partial^2 \mathbf{M}}{\partial t^2} \right\|_{L^1(0,T;L^2)} \tag{4.54}$$

where we have set

$$\dot{U}(t) = \frac{1}{2} \int_{\mathbb{R}} \left(\varepsilon_0 \left| \frac{\partial \mathbf{E}}{\partial t} \right|^2 + \mu_0 \left| \frac{\partial \mathbf{H}^*}{\partial t} \right|^2 \right) dx$$

Using Maxwell's equations we get :

$$\|\text{curl } \mathbf{H}^*\|_{L^\infty(0,T;L^2)} = \varepsilon_0 \left\| \frac{\partial \mathbf{E}}{\partial t} \right\|_{L^\infty(0,T;L^2)}$$

which gives :

$$\|\text{curl } \mathbf{H}^*\|_{L^\infty(0,T;L^2)} \leq \sqrt{2\varepsilon_0} \dot{U}^{\frac{1}{2}}(0) + \sqrt{\varepsilon_0 \mu_0} \left\| \frac{\partial^2 \mathbf{M}}{\partial t^2} \right\|_{L^1(0,T;L^2)} \tag{4.55}$$

Finally, from the equality (cf remark 1) :

$$H_x^* + M_x = \mathbf{H}_{0,x} + \mathbf{M}_{0,x}$$

we deduce

$$\|H_x^*\|_{L^\infty(0,T;L^\infty)} \leq 2\,\|\mathbf{M}_0\|_{L^\infty} + \|\mathbf{H}_{0,x}\|_{L^\infty} \tag{4.56}$$

Regrouping (4.52), (4.53), (4.55) and (4.56), we get

$$\begin{aligned} \|\mathbf{H}^*\|_{W(0,T))} \leq\ & \sqrt{\frac{2}{\mu_0}} U^{\frac{1}{2}}(0) + \left(\sqrt{\frac{2}{\mu_0}} + \sqrt{2\varepsilon_0} \right) \dot{U}^{\frac{1}{2}}(0) + 2\,\|\mathbf{M}_0\|_{L^\infty} + \|\mathbf{H}_{0,x}\|_{L^\infty} \\ & + \left\| \frac{\partial \mathbf{M}}{\partial t} \right\|_{L^\infty(0,T;L^2)} + (1 + \sqrt{\varepsilon_0 \mu_0}) \left\| \frac{\partial^2 \mathbf{M}}{\partial t^2} \right\|_{L^\infty(0,T;L^2)} \end{aligned} \tag{4.57}$$

Now to conclude we remark that :

$$\begin{cases} U^{\frac{1}{2}}(0) \leq \left(\sqrt{\frac{\varepsilon_0}{2}} + \sqrt{\frac{\mu_0}{2}} \right) R \\ 2\,\|\mathbf{M}_0\|_{L^\infty} + \|\mathbf{H}_{0,x}\|_{L^\infty} \leq 3R \end{cases} \tag{4.58}$$

while :

$$\dot{U}(0) = \frac{1}{2}\varepsilon_0 \left\| \frac{\partial \mathbf{E}}{\partial t}(\cdot,0) \right\|^2_{L^2} + \frac{1}{2}\mu_0 \left\| \frac{\partial \mathbf{H}}{\partial t}(\cdot,0) \right\|^2_{L^2} \tag{4.59}$$

$$= \frac{1}{2\varepsilon_0} \|\text{curl } \mathbf{H}_0\|^2_{L^2} + \frac{1}{2}\mu_0 \left\| \text{curl } \mathbf{E}_0 + \frac{\partial \mathbf{M}}{\partial t}(\cdot,0) \right\|^2_{L^2} \tag{4.60}$$

which yields since $\left| \frac{\partial \mathbf{M}}{\partial t} \right|^2 = \frac{\gamma}{1+\alpha^2} |\mathbf{H}_T \times \mathbf{M}|^2$

$$\dot{U}(0) \leq \left(\frac{1}{2\varepsilon_0} + \mu_0 \right) R^2 + \mu_0 \gamma \|\mathbf{M}_0\|^2_{L^\infty} \|\mathbf{H}_T(\cdot,0)\|_{L^2} \tag{4.61}$$

$$\leq \left(\frac{1}{2\varepsilon_0} + \mu_0 \right) R^2 + \mu_0 \gamma R^2 \|\mathbf{H}_T(\cdot,0)\|_{L^2} \tag{4.62}$$

From $\mathbf{H}_T(\cdot,0) = \mathbf{H}_0 + \mathbf{H}_s - KP(\mathbf{M}_0)$, we get

$$\|\mathbf{H}_T(\cdot,0)\|^2_{L^2} \leq 2\left(\|\mathbf{H}_0\|^2_{L^2} + \|K\|^2_{L^\infty} \|\mathbf{M}_0\|^2_{L^2} + \|\mathbf{H}_s\|^2_{L^2} \right) \tag{4.63}$$

$$\leq 2\|\mathbf{H}_s\|^2_{L^2} + 2\left(1 + \|K\|^2_{L^\infty}\right) R^2 \tag{4.64}$$

Thus

$$\dot{U}(0) \leq \left(\frac{1}{2\varepsilon_0} + \mu_0 + 2\gamma\mu_0 \|\mathbf{H}_s\|^2_{L^2} \right) R^2 + 2\gamma\mu_0 \left(1 + \|K\|^2_{L^\infty}\right) R^4$$

and then

$$\dot{U}^{\frac{1}{2}}(0) \leq \left(\frac{1}{2\varepsilon_0} + \mu_0 + 2\gamma\mu_0 \|\mathbf{H}_s\|^2_{L^2} \right)^{\frac{1}{2}} R + \left[2\gamma\mu_0 \left(1 + \|K\|^2_{L^\infty}\right) \right]^{\frac{1}{2}} R^2 \tag{4.65}$$

From (4.58) and (4.65), it is easy to show that there are two constants τ and σ depending only on ε_0, μ_0, α, γ, K and $\mathbf{H}_s$ such that :

$$\sqrt{\frac{2}{\mu_0}} U^{\frac{1}{2}}(0) + \left(\sqrt{\frac{2}{\mu_0}} + \sqrt{\frac{\mu_0}{2}} \right) \dot{U}^{\frac{1}{2}}(0) + 2\|\mathbf{M}_0\|_{L^\infty} + \|\mathbf{H}_{0,x}\|_{L^\infty} \leq \tau R + \sigma R^2 \tag{4.66}$$

Thanks to (4.57), this concludes the proof of theorem 5.

References

[1] E.A. Coddington , N. Levinson. *Theory of ordinary differential equation.* Macgraw-Hill, 1981.

[2] H. Cartan. *Cours de calcul différentiel.* Herman, 1966.

[3] P. Joly and O. Vacus. Mathematical and numerical studies of 1d non linear ferromagnetic materials. *soumis à M2AN*, 1996.

[4] P. Joly and O. Vacus. Propagation d'ondes en milieu ferromagnétique 1d : existence et unicité de solutions faibles. *Compte Rendu de l'Académie des Sciences*, 1996.

[5] A. Visintin. On landau-lifchitz' equations for ferromagnetism. *Japan journal of applied mathematics*, 2:69–84, 1985.

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